Announcements:
- PS6 due now
- Final Preflight posted, due next Wednesday noon fun, optional, easy bonus points
- **ICES**! please don’t skip written comments
- *No class this Friday* – woo hoo!

Last time:
- embraced $\Lambda$CMB cosmology *Q: what’s that?*
  *Q: examples of viable alternatives?*
- began move beyond linear perturbations
  *Q: why is this important? why is it hard?*
- Spherical collapse model *Q: what’s that?*
  *Q: qualitative results? quantitative results?*
Spherical Collapse: Quantitative Lessons

first-order pert: $\delta_{\text{lin}}(t) \propto D(t) \propto t^{2/3} \propto a_{\text{bg}}$

same as usual linear result!

$$\delta_{\text{nonlin}}(t) = \left( \frac{a_{\text{bg}}}{a_{\text{nonlin}}} \right)^3 - 1$$  \hspace{1cm} (1)

$$\delta_{\text{lin}}(t) \approx \frac{3}{20} \left( \frac{12\pi t}{t_{\text{coll}}} \right)^{2/3}$$  \hspace{1cm} (2)

connects full nonlinear result with linear counterpart
→ maps between the two

E.g., at turnaround
$$\delta_{\text{nonlin}} = \frac{(6\pi)^2}{4^3} = 5.6, \text{ but } \delta_{\text{lin}} = 1.06$$

at virialization (PS6):
$$\delta_{\text{nonlin}} \approx 180, \text{ but } \delta_{\text{lin}} = 1.69$$
→ defines a critical linear overdensity

Q: why useful?
Strategy: look at initial linear density field
find perturbations with linear growth \( \delta_{\text{lin}}(t) = D(t)\delta_i > 1.69 \)
→ these will be collapsed objects by time \( t \)

- \( \delta_c \) cut in *linearized* \( \delta_{\text{lin}}(t_0) \) divides virialized vs nonvirialized

- also: in nonlinear field, can use \( \delta_{\text{vir}} \sim 180 \)
as working *definition* of collapsed structure
good for comparing theory, observation *Q: procedure?*
Nonlinear Evolution: Lessons from Spherical Collapse

Qualitatively
▷ overdensity evolves as closed “subuniverse”
▷ starts expanding, but slower than cosmic background pulls away from Hubble flow: reach max expansion, then turnaround
▷ virialize → form bound object
▷ no further expansion, except due to accretion, merging

Quantitatively
▷ can compute both $\delta_{\text{lin}}(t)$ and exact $\delta(t)$
  gives mapping from easy to (more) correct
▷ collapse/virialization when $\delta_{\text{lin}} = 1.69$ and $\delta = 18\pi^2 \approx 180$
  recipe for forecasting structures in initial field $\delta_{\text{init}} \ll 1$
  recipe for defining halos: region surrounding density peak and having overdensity $\delta\rho/\rho \sim 180$
★ Given these, can devise analytical tools to describe distribution of structures
Press-Schechter Analysis

Outlook
adopt hierarchical picture (i.e., some form of CDM)
⇒ matter at every point belongs to some structure
over time: go from many small structures to fewer, larger ones

Goal
Given properties of density field—i.e., \( P_{\text{init}}(k) \) and \( P(k, t) = T_k^2(t)P_{\text{init}}(k) \)
Compute distribution of structures as function of mass, time

Quantitatively: want “mass function”
comoving number density of structures in mass range \((M, M + dM)\):

\[
\frac{dn_{\text{com}}}{dM}(M, t)
\]  

(3)

from this, can compute many other things
e.g., density in \((M, M + dM)\) Q: which is...?
Press-Schechter Ingredients/Assumptions

• given mass $M$, filter density field on comov length $R$ such that $M = \frac{4\pi}{3} \rho_{bg,\text{com}}(t) R^3$

density contrast has variance $\sigma^2(M) = \int P(k) \ W(k; R) \ d^3k$

• in linear regime, density field obeys Gaussian statistics:
in filtered field, probability of finding contrast in $(\delta_{\text{lin}}, \delta_{\text{lin}} + d\delta_{\text{lin}})$:

$$P(\delta_{\text{lin}}; M, t) \ d\delta_{\text{lin}} = \frac{1}{\sqrt{2\pi\sigma^2(M, t)}} \exp \left[ -\frac{\delta_{\text{lin}}^2}{2\sigma^2(M, t)} \right] \ d\delta_{\text{lin}}$$

(4)

why only good in linear regime Q: why?

• Spherical collapse model maps from linear $\rightarrow$ nonlinear identifies linear contrast threshold $\delta_c \approx 1.69$ for collapsed objects

note: $\delta_c$ is time indep! (in EdS cosmo)

$\Rightarrow$ can find fraction of cosmic mass in objects of mass $M$

Q: how?
fraction of mass or of comoving volume in collapsed objects of mass \( M \) at time \( t \) is

\[
f(> \delta_c; M, t) = \int_{\delta_c}^{\infty} P(\delta_{\text{lin}}; M, t) \, d\delta_{\text{lin}}
\]

\[
= \frac{1}{\sqrt{2\pi}\sigma^2(M, t)} \int_{\delta_c}^{\infty} \exp \left[-\frac{\delta_{\text{lin}}^2}{2\sigma^2(M, t)}\right] \, d\delta_{\text{lin}}
\]

\[
= \frac{1}{\sqrt{2\pi}} \int_{\delta_c/\sqrt{2\sigma}}^{\infty} e^{-u^2} \equiv \frac{1}{2} \text{erfc} \left[ \frac{\delta_c}{\sqrt{2\sigma}(M, t)} \right]
\]

• for realistic \( P(k) \), \( \sigma^2(M) \sim \int k^3 P(k)W(M)dk/k \sim M^{-(n+3)/3} \)
  → at fixed mass, \( \sigma(M, t) \) monotonically decreases with \( M \) (down to some minimum \( M \) cutoff)
• \( \sigma(M, t) \) evolves (linearly) as \( \sigma \sim a(t) \sim 1/(1 + z) \)

\[ Q: \text{implications for mass distribution at fixed time?} \]
\[ Q: \text{implications for structure formation over time?} \]
Press-Schechter: mass fraction and structure formation

\[ f(> \delta_c; M, t) = \frac{1}{\sqrt{2\pi}} \int_{\delta_c/\sqrt{2}\sigma}^{\infty} e^{-u^2} \, du = \frac{1}{2} \text{erfc} \left( \frac{\delta_c}{\sqrt{2}\sigma(M, t)} \right) \]  

\( \star \) mass distribution at fixed \( t \):

- as filter mass \( M \) decreases, variance \( \sigma(M) \) increases
- more large fluctuations \( \rightarrow \) more above threshold
- more structures at smaller masses
  - i.e., \( \delta_c/\sqrt{2}\sigma(M) \) decreases \( \rightarrow \) larger \( f \)
- \( \Rightarrow \) smallest halos most numerous \( \rightarrow \) hierarchy of masses!

\( \star \) time evolution at fixed \( M \):

- at time, scale factor increases, variance \( \sigma(t) \propto a(t) \) increases
- more structures at fixed mass
- small structures merge \( \rightarrow \) larger (at expense of smallest)
- \( \Rightarrow \) hierarchical clustering!
Press & Schechter (1974):

Note that structures can only be made from *overdensities* but *underdensities* (voids) occupy mass fraction $f(\delta_{\text{lin}} < 0) = 1/2$

so fraction of *overdensities* in collapsed objects of $M$ is

$$F(> \delta_c; M, t) = \frac{f(\delta_{\text{lin}} > \delta_c)}{f(\delta_{\text{lin}} > 0)} = 2f(\delta_{\text{lin}} > \delta_c)$$

(9)

famous factor of two!

Compare mass fraction at $M$ and $M + dM$: difference

$$dF = F(M + dM) - F(M) \simeq \frac{dF}{dM} dM$$

(10)

$$= \sqrt{\frac{2}{\pi}} \frac{d\sigma(M)^{-1}}{dM} \frac{\delta_c}{\sigma(M)} e^{-\delta_c^2/2\sigma^2(M)} dM$$

(11)
But probability of finding structure $M$ in filter volume $V_{\text{com}} = M/\rho_{\text{bg}}$ is

$$dF(M) = V \frac{dn}{dM} dM = \frac{M}{\rho_{\text{bg}}} \frac{dn}{dM} dM$$  \hspace{1cm} (12)$$

and so PS mass function is

$$M \frac{dn}{dM} = \frac{\rho_{\text{bg}}}{M} M \frac{dF}{dM} = \sqrt{\frac{2}{\pi}} \frac{d \ln \sigma(M)^{-1}}{d \ln M} \frac{\delta_c}{\sigma(M)} \frac{\rho_{\text{bg}}}{M} e^{-\delta_c^2/2\sigma^2(M)}$$

- implicitly also a function of $t$ via $\rho_{\text{bg}}(t)$ and $\sigma(M,t)$
- encodes and quantifies hierarchical clustering

from this can immediately find, e.g., distribution of (comoving) density across masses of collapsed objects:

$$\frac{d\rho(M)}{dM} = M \frac{dn}{dM}$$  \hspace{1cm} (13)$$
Press-Schechter: Summary

Quantitative Output
★ Easy to use, very powerful (semi-)analytic mass function

Qualitative Worldview/Limitations
★ every point lies in exactly one structure:
  largest above threshold
★ all structures have $\delta_{\text{lin}} = \delta_c$: born today!
★ PS blind to interior substructure
  and formation history of a given object

Q: how to test PS theory?
Q: which structures should be best described? worst?
Tests of Press-Schechter

Versus Numerical Simulations
PS is idealized analytic approximation of hierarchical clustering assumes true density field $\delta$ perfectly mapped onto linear field $\delta_{\text{lin}}$ vis spherical collapse model

Even if underlying CDM, hierarchy idea right, PS approximate $\rightarrow$ test against numerical simulations w/ non-ideal $\delta$ field
results: unreasonably good agreement!

Versus Observations
Best applicable to those just formed: $\sigma(R) \sim \sigma_8 \sim 1$
$\rightarrow$ galaxy clusters! $M \sim 10^{15} \ M_\odot$, and so PS gives

$$n(M) \sim M \frac{dn}{dM} \sim \frac{\rho_0}{M} \nu e^{-\nu^2/2} \sim \frac{\rho_0}{M} \sim 10^{-4} \ \text{Mpc}^{-3}$$

about right! (where $\nu = \frac{\delta_c}{\sqrt{2}\sigma} \sim 1$)
...and works unreasonably well at other scales too
Director’s Cut Extras
Press-Schechter II: Excursion Sets

More sophisticated (and insightful) derivation of same result

Sketch of procedure:
1. given initial density field and (Gaussian) filter window
2. pick a point $\vec{x}$ in space, filter over neighborhood $R$, mass $M(R)$
3. scan down in mass: at $M \rightarrow \infty$, $\sigma(M) \rightarrow 0$ Q: why?
   and so filtered $\delta(\vec{x})_M = 0$
3. as $M$ decreases, $\sigma(M)$ increases
   filtered $\delta(\vec{x})_M \neq 0$, alternates sign, amplitude
   $\Rightarrow \delta(\vec{x})_M$ is a random walk vs $\sigma(M)$! exactly!
4. can ask: at which $M$ does $\delta(\vec{x})_M$ first cross threshold $\delta_c$
   $\Rightarrow$ this sets $M$ of structure containing point $\vec{x}$
5. repeat for all $\vec{x}$ and average $\Rightarrow$ PS distribution follows!

Q: limitations/implicit assumptions?