Astro 596/496 PC
Lecture 39
April 28, 2010

Announcements:
- PS6 due now
- Final Preflight posted, due next Wednesday noon
  fun, optional, easy bonus points
- **ICES**! please don't skip written comments
- *No class this Friday* – woo hoo!

Last time:
- embraced ΛCMB cosmology *Q: what's that?*
  *Q: examples of viable alternatives?*
- began move beyond linear perturbations
  *Q: why is this important? why is it hard?*
  Spherical collapse model *Q: what's that?*
  *Q: qualitative results? quantitative results?*

# Spherical Collapse: Quantitative Lessons

first-order pert: $\delta_{\rm lin}(t) \propto D(t) \propto t^{2/3} \propto a_{\rm bg}$
same as usual linear result!

$$\delta_{\rm nonlin}(t) = \left(\frac{a_{\rm bg}}{a_{\rm nonlin}}\right)^3 - 1 \qquad (1)$$

$$\delta_{\rm lin}(t) \approx \frac{3}{20}\left(\frac{12\pi t}{t_{\rm coll}}\right)^{2/3} \qquad (2)$$

connects full nonlinear result with linear counterpart
$\rightarrow$ maps between the two

E.g., at turnaround
$\delta_{\rm nonlin} = (6\pi)^2/4^3 = 5.6$, but $\delta_{\rm lin} = 1.06$
at virialization (PS6):
$\delta_{\rm nonlin} \approx 180$, but $\delta_{\rm lin} = 1.69$
$\rightarrow$ defines a critical linear overdensity
*Q: why useful?*

Strategy: look at initial linear density field
find perturbations with linear growth $\delta_{\mathrm{lin}}(t) = D(t)\delta_i > 1.69$
$\longrightarrow$ these will be collapsed objects by time $t$

- $\delta_c$ cut in *linearized* $\delta_{\mathrm{lin}}(t_0)$ divides virialized vs nonvirialized

- also: in nonlinear field, can use $\delta_{\mathrm{vir}} \sim 180$
  as working *definition* of collapsed structure
  good for comparing theory, observation *Q: procedure?*

# Nonlinear Evolution: Lessons from Spherical Collapse

Qualitatively

▷ overdensity evolves as closed "subuniverse"

▷ starts expanding, but slower than cosmic background
pulls away from Hubble flow: reach max expansion, then turnaround

▷ virialize $\rightarrow$ form bound object

▷ no further expansion, except due to accretion, merging

Quantitatively

▷ can compute both $\delta_{\rm lin}(t)$ and exact $\delta(t)$
gives mapping from *easy* to (more) *correct*

▷ collapse/virialization when $\boxed{\delta_{\rm lin} = 1.69 \text{ and } \delta = 18\pi^2 \simeq 180}$
recipe for forecasting strucutres in initial field $\delta_{\rm init} \ll 1$
recipe for defining halos: region surrounding density peak
and having overdensity $\delta\rho/\rho \sim 180$



⋆ Given these, can devise analytical tools to describe distribution of structures

# Press-Schechter Analysis

Outlook
adopt hierarchical picture (i.e., some form of CDM)
$\Rightarrow$ matter at *every* point belongs to some structure
over time: go from many small structures to fewer, larger ones

Goal
Given properties of density field–i.e., $P_{\mathrm{init}}(k)$ and $P(k,t) = T_k^2(t) P_{\mathrm{init}}(k)$
Compute distribution of structures as function of mass, time

Quantitatively: want "mass function"
comoving number density of structures
in mass range $(M, M + dM)$:

$$\frac{dn_{\mathrm{com}}}{dM}(M,t) \tag{3}$$



from this, can compute many other things
e.g., density in $(M, M + dM)$ *Q: which is...?*

# Press-Schechter Ingredients/Assumptions

• given mass $M$, filter density field
on comov length $R$ such that $M = 4\pi/3\ \rho_{\text{bg,com}}(t) R^3$
density contrast has variance $\sigma^2(M) = \int P(k)\ W(k;R)\ d^3k$

• in linear regime, density field obeys Gaussian statistics:
in filtered field, probability of finding contrast in $(\delta_{\text{lin}}, \delta_{\text{lin}} + d\delta_{\text{lin}})$:

$$P(\delta_{\text{lin}}; M, t)\ d\delta_{\text{lin}} = \frac{1}{\sqrt{2\pi\sigma^2(M,t)}} \exp\left[-\frac{\delta_{\text{lin}}^2}{2\sigma^2(M,t)}\right]\ d\delta_{\text{lin}} \qquad (4)$$

why only good in linear regime *Q: why?*

• Spherical collapse model maps from linear → nonlinear
identifies *linear contrast threshold* $\delta_c \simeq 1.69$ for collapsed objects
  note: $\delta_c$ is time indep! (in EdS cosmo)
⇒ can find fraction of cosmic mass in objects of mass $M$
*Q: how?*

fraction of mass or of comoving volume
in collapsed objects of mass $M$ at time $t$ is

$$
\begin{aligned}
f(>\delta_c; M, t) &= \int_{\delta_c}^{\infty} P(\delta_{\rm lin}; M, t)\ d\delta_{\rm lin} && (5)\\
&= \frac{1}{\sqrt{2\pi\sigma^2(M,t)}} \int_{\delta_c}^{\infty} \exp\left[-\frac{\delta_{\rm lin}^2}{2\sigma^2(M,t)}\right]\ d\delta_{\rm lin} && (6)\\
&= \frac{1}{\sqrt{2\pi}} \int_{\delta_c/\sqrt{2}\sigma}^{\infty} e^{-u^2} \equiv \frac{1}{2}{\rm erfc}\left[\frac{\delta_c}{\sqrt{2}\sigma(M,t)}\right] && (7)
\end{aligned}
$$

- for realistic $P(k)$, $\sigma^2(M) \sim \int k^3 P(k) W_k(M) dk/k \sim M^{-(n+3)/3}$
  $\rightarrow$ at fixed mass, $\sigma(M,t)$ monotonically decreases with $M$
  (down to some minimum $M$ cutoff)
- $\sigma(M,t)$ evolves (linearly) as $\sigma \sim a(t) \sim 1/(1+z)$



*Q: implications for mass distribution at fixed time?*
*Q: implications for structure formation over time?*

## Press-Schechter: mass fraction and structure formation

$$f(>\delta_c; M, t) = \frac{1}{\sqrt{2\pi}} \int_{\delta_c/\sqrt{2}\sigma}^{\infty} e^{-u^2} = \frac{1}{2}\mathrm{erfc}\left[\frac{\delta_c}{\sqrt{2}\sigma(M,t)}\right] \qquad (8)$$

⋆ mass distribution at fixed $t$:

as filter mass $M$ decreases, variance $\sigma(M)$ increases

▷ more large fluctuations → more above threshold

▷ more structures at smaller masses

i.e., $\delta_c/\sqrt{2}\sigma(M)$ decreases → larger $f$

⇒ smallest halos most numerous → hierarchy of masses!

⋆ time evolution at fixed $M$:

at time, scale factor increases, variance $\sigma(t) \propto a(t)$ increases

▷ more structures at fixed mass



▷ small structures merge → larger (at expense of smallest)

⇒ hierarchical clustering!

# Press-Schechter Mass Function I: Quick-n-Dirty

Press & Schechter (1974):
note that structures can only be made from *over*densities
but *under*densities (voids) occupy mass fraction $f(\delta_{\rm lin} < 0) = 1/2$
so fraction of *overdensites* in collapsed objects of $M$ is

$$F(>\delta_c; M, t) = \frac{f(\delta_{\rm lin} > \delta_c)}{f(\delta_{\rm lin} > 0)} = 2 f(\delta_{\rm lin} > \delta_c) \tag{9}$$

famous factor of two!

Compare mass fraction at $M$ and $M + dM$: difference

$$dF = F(M + dM) - F(M) \simeq \frac{dF}{dM}\, dM \tag{10}$$

$$= \sqrt{\left(\frac{2}{\pi}\right)} \frac{d\sigma(M)^{-1}}{dM} \frac{\delta_c}{\sigma(M)}\, e^{-\delta_c^2/2\sigma^2(M)}\, dM \tag{11}$$

But probability of finding structure $M$ in filter volume $V_{\rm com} = M/\rho_{\rm bg}$ is

$$dF(M) = V\ \frac{dn}{dM}\ dM = \frac{M}{\rho_{\rm bg}}\frac{dn}{dM}\ dM \tag{12}$$

and so PS mass function is

$$M\frac{dn}{dM} = \frac{\rho_{\rm bg}}{M}\ M\frac{dF}{dM} = \sqrt{\frac{2}{\pi}}\ \frac{d\ln\sigma(M)^{-1}}{d\ln M}\ \frac{\delta_c}{\sigma(M)}\ \frac{\rho_{\rm bg}}{M}\ e^{-\delta_c^2/2\sigma^2(M)}$$

- implicitly also a function of $t$ via $\rho_{\rm bg}(t)$ and $\sigma(M,t)$
- encodes and quantifies hierarchical clustering

from this can immediately find, e.g., distribution of (comoving) density across masses of collapsed objects:

$$\frac{d\rho(M)}{dM} = M\frac{dn}{dM} \tag{13}$$

# Press-Schechter: Summary

## Quantitative Output

⋆ Easy to use, very powerful (semi-)analytic mass function

## Qualitative Worldview/Limitations

⋆ *every* point lies in exactly one structure:
largest above threshold

⋆ all structures have $\delta_{\rm lin} = \delta_c$: born today!

⋆ PS blind to interior substructure
and formation history of a given object

*Q: how to test PS theory?*

*Q: which structures should be best described? worst?*

# Tests of Press-Schechter

## Versus Numerical Simulations

PS is idealized analytic approximation of hierarchical clustering
assumes true density field $\delta$ perfectly mapped onto
linear field $\delta_{\mathrm{lin}}$ vis spherical collapse model

Even if underlying CDM, hierarchy idea right, PS approximate
$\rightarrow$ test against numerical simulations w/ non-ideal $\delta$ field
results: unreasonably good agreement!

## Versus Observations

Best applicable to those just formed: $\sigma(R) \sim \sigma_8 \sim 1$
$\rightarrow$ galaxy clusters! $M \sim 10^{15}\ M_{\odot}$, and so PS gives

$$n(M) \sim M \frac{dn}{dM} \sim \frac{\rho_0}{M} \nu e^{-\nu^2/2} \sim \frac{\rho_0}{M} \sim 10^{-4}\ \mathrm{Mpc}^{-3} \qquad (14)$$

about right! (where $\nu = \delta_c/\sqrt{2}\sigma \sim 1$)
...and works unreasonably well at other scales too

# Director’s Cut Extras

# Press-Schechter II: Excursion Sets

More sophisticated (and insightful) derivation of same result

Sketch of procedure:
1. given initial density field and (Gaussian) filter window
2. pick a point $\vec{x}$ in space, filter over neighborhood $R$, mass $M(R)$
3. scan down in mass: at $M \to \infty$, $\sigma(M) \to 0$ *Q: why?*
   and so filtered $\delta(\vec{x})_M = 0$
3. as $M$ decreases, $\sigma(M)$ increases
   filtered $\delta(\vec{x})_M \neq 0$, alternates sign, amplitude
   $\Rightarrow \delta(\vec{x})_M$ is a random walk vs $\sigma(M)$! exactly!
4. can ask: at which $M$ does $\delta(\vec{x})_M$ first cross threshold $\delta_c$
   $\Rightarrow$ this sets $M$ of structure containing point $\vec{x}$
5. repeat for all $\vec{x}$ and average $\to$ PS distribution follows!

*Q: limitations/implicit assumptions?*