Announcements:
• PS1 due next Friday, Feb. 5
  ▶ Director’s Cut Extras today: magnitude scale
• Office Hours: 3–4 pm Thursday, or by appointment
  note phase correlation with Friday due date
• Preflight 1 was due today—thanks!

Last time: Cosmodynamics I—Newtonian Cosmology
result: the right answer—Dr. Friedmann’s famous equation
Suitable for framing, T-shirts, tattoos...

Q: what’s the Friedmann eq? who cares—i.e., why is it useful?
Q: in Friedmann—what’s a parameter? what’s a variable?
Friedmann (Energy) Equation

\[ H^2 = \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi}{3} G \rho - \frac{\kappa c^2}{R^2 a^2} \]  

variables change with time
- \( a \): cosmic scale factor
- \( \rho \): total cosmic mass-energy density

parameters constant, fixed for all time
- \( \kappa = \pm 1 \) or 0: sign of “energy” (curvature) term
- \( R \): characteristic lengthscale, GR → curvature scale

Q: how does expansion of \( U \) depend on contents of \( U \)?
Q: how does expansion of \( U \) not depend on contents of \( U \)?

Q: what about acceleration—\( \ddot{a} \)?
Newtonian analysis gives $\ddot{a}$ for $P \to 0$

In full GR: with $P \neq 0$, get Friedmann acceleration eq.

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}G(\rho + 3P/c^2)$$ (2)

**Pressure and Friedmann**

★ in “energy” ($\dot{a}$) eq.: $P$ absent, even in full GR

★ in acceleration eq., GR $\to$ $P$ present, same sign as $\rho$

   adds to “active gravitational mass”

   Q: why? Q: contrast with hydrostatic equilibrium?

Friedmann energy eq is “equation of motion” for scale factor

   i.e., governs evolution of $a(t)$.

To solve, need to know how $\rho$ depends on $a$

Q: how figure this out?

Q: hint: what is $\rho(a)$ for non-rel matter?
Density Evolution: Matter

if cosmic matter is non-relativistic:
- particle speeds $v \ll c$, and/or $kT \ll mc^2$ (particle rest energy)
- mass is conserved

in comoving sphere with volume $V \propto a^3$, mass conservation gives:

$$dM = d(\rho V) \propto d(\rho a^3) = 0$$  \hspace{1cm} (3)

gives density

$$\rho_{\text{non-rel}} \propto \frac{1}{V} \propto a^{-3}$$ \hspace{1cm} (4)

definition: to cosmologist, matter $\equiv$ non-relativistic matter
today: $\rho_{\text{matter}}(t_0) \equiv \rho_{m,0}$
at other epochs (while still non-relativistic):

$$\rho_m = \rho_{m,0} a^{-3}$$ \hspace{1cm} (5)
Alternative Derivation: Fluid Picture

in fluid picture: mass conservation $\rightarrow$ continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$  \hspace{1cm} (6)

put $\rho = \rho(t)$ and $\vec{v} = H\vec{r}$:

$$\dot{\rho} + H\rho \nabla \cdot \vec{r} = \dot{\rho} + 3\frac{\dot{a}}{a}$$  \hspace{1cm} (7)

$$\frac{d\rho}{\rho} = -3\frac{da}{a}\rho$$  \hspace{1cm} (8)

$$\rho \propto a^{-3}$$  \hspace{1cm} (9)
A Matter-Only Universe

consider a universe containing only non-relativistic matter

Friedmann:

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho - \frac{\kappa c^2}{R^2} \frac{1}{a^2}
\]

(10)

\[
= \frac{8\pi G}{3} \rho_0 a^{-3} - \frac{\kappa c^2}{R^2} a^{-2}
\]

(11)

For \( \kappa = 0 \): “Einstein-de Sitter”

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho_0 a^{-3}
\]

(12)

evaluate today: \( H_0^2 = \frac{8\pi G \rho_0}{3} \)

\[
a^{1/2} da = H_0 dt
\]

(13)

\[
2/3 a^{3/2} = H_0 t
\]

(14)

Q: implicit assumptions in solution?
Einstein-de Sitter:

$$t = \frac{2}{3} a^{3/2} H_0^{-1}$$  \hspace{1cm} (15)

$$a = \left(\frac{3}{2} H_0 t\right)^{2/3} = \left(\frac{t}{t_0}\right)^{2/3}$$  \hspace{1cm} (16)

Now unpack the physics:

- boundary condition: \( a = 0 \) at \( t = 0 \) → “big bang”
- \( a \propto t^{2/3} \) Q: interpretation?
- \( H = 2/3 \) 1/t ≠ const “Hubble parameter” Q: interp?
- present age: \( t_0 = 2/3 \) \( H_0^{-1} = 2/3 \) \( t_H < t_H \) Q: interp?
- U. half its present age at \( a = 2^{-2/3} = 0.63 \)
- objects half present separation (and 8× more compressed) at \( t = 2^{-3/2} t_0 = 0.35 t_0 \)
- using measured value of \( H_0 \), calculate \( t_0 = 8.9 \) Gyr
  but know globular clusters have ages \( t_{gc} \gtrsim 12 \) Gyr Q: huh?
Matter and Curvature

What if $\kappa = 1$?

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho_0 a^{-3} - \frac{c^2}{R^2} a^{-2}
\]

$a$ cannot grow without bound $Q$: why?

$Q$: what is $a_{\text{max}}$?

$Q$: why are we sure that $U$ recollapses after $t(a_{\text{max}})$?

fate: collapse continues back to $a = 0$: “big crunch!”

What if $\kappa = -1$?

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho_0 a^{-3} + \frac{c^2}{R^2} a^{-2}
\]

$a$ grows without bound $Q$: why?

fate: expand forever—“big chill”

at large $t$, “curvature-dominated”: $a(t) \to ct/R$ $Q$: why?

$Q$: how can we tell what our $\kappa$ value is?
Geometry, Density, and Dynamics

rewrite Friedmann

\[ 1 = \frac{8\pi G\rho}{3H^2} - \frac{\kappa c^2}{R^2}(aH)^{-2} = \Omega - \frac{\kappa c^2}{R^2}(aH)^{-2} \]  
(17)

where the density parameter is

\[ \Omega = \frac{\rho}{\rho_{\text{crit}}} \]  
(18)

and the critical density is

\[ \rho_{\text{crit}} = \frac{3H^2}{8\pi G} \]  
(19)

Note: for a particular density component \( \rho_i \)

- corresponding density parameter is \( \Omega_i = \rho_i/\rho_{\text{crit}} \)

and thus total sums all species: \( \Omega \equiv \Omega_{\text{tot}} = \sum_i \Omega_i \)
Note that
\[ \kappa = \left( \frac{aHR}{c} \right)^2 (\Omega - 1) = (\text{pos def}) \times (\Omega - 1) \]

fate* (and geometry) of Universe \( \Leftrightarrow \kappa \Leftrightarrow \Omega - 1 \)

if \( \Omega = 1 \) ever:
bullet \( \Omega = 1 \) always; \( \kappa = 0 \) \( \rightarrow \) expand forever

if \( \Omega < 1 \) ever:
bullet \( \Omega < 1 \) always; \( \kappa = -1 \) \( \rightarrow \) expand forever

if \( \Omega > 1 \) ever:
bullet \( \Omega > 1 \) always; \( \kappa = +1 \) \( \rightarrow \) recollapse

Q: but if \( \Omega \) just a stand-in for \( \kappa \), why useful?

*\( \kappa \) always gives geometry, but \( \kappa \) and fate decoupled if \( \Lambda \neq 0 \)
can determine $\Omega \propto \rho / H^2$
from *locally measurable quantities* $\rho$ and $H$:
\[ \rightarrow \text{cosmic fate & geometry knowable!} \]
\[ \ldots \text{and become *experimental* questions!} \]

But recall:
so far, only have considered non-relativistic matter
definitely an incomplete picture
\[ \rightarrow \text{at minimum, must include photons!} \]
Director’s Cut Extras: The Magnitude Scale
Star Brightness: Magnitudes

star brightness (flux) measured in **magnitude** scale
magnitude = “rank” : smaller \( m \) → **brighter**, more flux
Sorry.

Magnitudes use a **logarithmic** scale:
- difference of 5 mag is factor of 100 in flux:
  \[ m_2 - m_1 = -2.5 \log_{10} \frac{F_2}{F_1} \] (definition of mag scale!)
- mag units: dimensionless! (but usually say “mag”)
  since always a log of ratio of two dimensionful
  fluxes with physical units like \( \text{W}/\text{m}^2 \)

What is mag **difference** \( m_2 - m_1 \):
Q: if \( F_2 = F_1 \)?
Q: what is sign of difference if \( F_2 > F_1 \)?
Q: for equidistant light bulbs, \( L_1 = 100\text{Watt}, L_2 = 50\text{Watt} \)?
Apparent Magnitude

a measure of star flux = (apparent) brightness

- no distance needed
- arbitrary mag zero point set for convenience:
  historically: use bright star Vega: \( m(\text{Vega}) \equiv 0 \)
  then all other mags fixed by ratio to Vega flux

- ex: Sun has **apparent** magnitude \( m_\odot = -26.74 \)
  i.e., \( -2.5 \log_{10}(F_\odot/F_{\text{Vega}}) = -26.74 \)
  so \( F_{\text{Vega}} = 10^{-26.74/2.5} F_\odot = 2 \times 10^{-11} F_\odot \)
- ex: Sirius has \( m_{\text{Sirius}} = -1.45 \rightarrow \text{brighter} \) than Vega
  so: \( F_{\text{Sirius}} = 3.8 F_{\text{Vega}} = 8 \times 10^{-11} F_\odot \)
- ex: \( m_{\text{Polaris}} = 2.02 \) Q: rank Polaris, Sirius, Vega?
if distance to a star is known can also compute **Absolute Magnitude**

abs mag $M = \text{apparent mag if star placed at } d_0 = 10 \text{ pc}$

*Q: what does this measure, effectively?*
**Absolute Magnitude**

absolute magnitude $M = \text{apparent mag at } d_0 = 10 \text{ pc}$

places all stars at constant fixed distance
→ a stellar “police lineup”
→ then differences in $F$ only due to diff in $L$
→ absolute mag effectively measure luminosity

Sun: abs mag $M_\odot = 4.76$ mag
Sirius: $M_{\text{Sirius}} = +1.43$ mag
Vega: $M_{\text{Vega}} = +0.58$ mag
Polaris: $M_{\text{Polaris}} = -3.58$ mag
$\epsilon$ Eridani: $M_{\epsilon\text{Eri}} = +6.19$ mag  (nearest exoplanet host; $d = 3.2 \text{ pc}$)

**Q:** rank them in order of descending $L$?

Immediately see that Sun neither most nor least luminous star around
**Distance Modulus**

take ratio of actual star flux vs “lineup” flux at abs mag distance \(d_0 = 10 \text{ pc}\):

\[
\frac{F}{F_0} = \frac{L/4\pi d^2}{L/4\pi d_0^2}
\]

(20)

which, after simplification, leads to

\[
m - M = 5 \log \left( \frac{d}{10 \ \text{pc}} \right)
\]

(21)

- depends only on distance \(d\), not on luminosity!
  - can use as measure of distance
- \(m - M \equiv \text{“distance modulus”}\), sometimes denoted \(\mu\)