Astro 596/496 PC

Lecture 8

Feb. 5, 2010

Announcements:

- PS1 due now!
- PF2 due next Friday noon
- High-Energy Seminar next Monday, 3pm, Loomis 464:
  Dan Bauer (Fermilab)
  "Recent Results from CDMS" – dark matter hint?!

Last time:

- $\Omega_0 = 1$! but $\Omega_{\mathrm{matter}} = 0.27$!?
- end of the road for Newtonian Cosmology:
  this situation requires General Relativity!
- *Q: events? spacetime?*

# Spacetime Coordinates

Each event specifies a unique point in
spacetime = collection of all events

lay down coordinate system: 3 space coords, one time
4-dimensional, but as yet time & space always "orthogonal"

e.g., time $t$, Cartesian $x, y, z$: event $\rightarrow (t, x, y, z)$
physics asks (and answers) what is the relationship
between two events, e.g., $(t_1, x_1, y_1, z_1)$ and $(t_2, x_2, y_2, z_2)$

# Pre-Relativity: Aristotle

$x, y, z$ Cartesian (Euclidean geometry)
spatial distance $\ell$ between events is:

$$\ell^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \tag{1}$$

and is independent of time
elapsed time between events is: $t_2 - t_1$
and is independent of space
"absolute space" and "absolute time"

Is a particle at rest? $\Leftrightarrow$ do $(x, y, z)$ change?
$\longrightarrow$ "absolute rest, absolute motion"

ω *Diagram: Aristotelian spacetime*
unique "stacking" of "time slices"

# Life According to Aristotle

Note: even in absolute space
event location indep of coordinate description
e.g., two observers choose coordinates different by a rotation:
$(x, y)$ and $(x', y') = (x \cos\theta - y \sin\theta, y \cos\theta + x \sin\theta)$
preserves distance from origin: $x^2 + y^2 = (x')^2 + (y')^2$

objects (observers) at rest:
same $x, y, z$ always, $t$ ticks forward
geometrically, a line in spacetime: **"world line"**
if at rest: world line vertical
constant speed: $x = vt$: diagonal line

light: moves at "speed of light" $c$
$\longrightarrow$ well-defined, since motion absolute
in spacetime: light pulse at origin $(t, x, y, z) = (0, 0, 0, 0)$
moves so that distance $\ell = \sqrt{x^2 + y^2 + z^2} = ct$
geometrically: **light cone**

## Galilean Relativity

consider two identical laboratories
(same apparatus, scientists, funding, vending machines)
move at constant velocity wrt each other

Galileo:
no experiment either can do (without looking outside)
will answer "which lab is moving"
$\rightarrow$ *no absolute motion*, only relative velocity

Newton: laws of mechanics invariant
for observers moving at const v
"inertial observers"

Implications for spacetime
no absolute motion $\rightarrow$ *no absolute space*
(but still no reason to abandon absolute time)

# Galilean Frames

each inertial obs has own personal frame:
obs (“Angelina”) at rest in own frame: $(x, y, z)$ same for all $t$
but to another obs (“Brad”) in relative motion $\vec{v} = v\hat{x}$
B sees A’s frame as time-dependent:

$$x_{\text{A as seen by B}} = x' = x - vt \tag{2}$$

but still absolute time: $t' = t$
Newton’s laws (and Gravity) hold in both frames
can show: $d^2\vec{x}/dt^2 = \vec{F}(\vec{x}) \Rightarrow d^2\vec{x}'/dt'^2 = \vec{F}(\vec{x}')$
“Galilean invariance”

Geometrically:
different inertial frames $\rightarrow$ transformation of spacetime
slide the “space slices” at each time
(picture “shear,” or beveling a deck of cards)

# Trouble for Galileo

Maxwell: equations govern light
very successful, but:

- predicts unique (constant) light speed $c$–relative to whom?
- Maxwell eqs **not** Galilean invariant

Lorentz: Maxwell eqs invariant when

$$
\begin{aligned}
t' &= \gamma(t - vx/c^2) \quad (3)\\
x' &= \gamma(x - vt) \quad (4)\\
y' &= y \quad (5)\\
z' &= z \quad (6)
\end{aligned}
$$

where $\gamma = 1/\sqrt{1 - v^2/c^2} \geq 1$

Einstein:
Lorentz transformation not just a trick
but correct relationship between inertial frames!
$\Rightarrow$ this is the way the world is

# Einstein: Special Relativity

consider two events

$(t, x, y, z)$ and $(t + \Delta t, x + \Delta x, y + \Delta y, z + \Delta z)$

diff inertial obs disagree about $\Delta t$ *and* $\Delta \vec{x}$

but all *agree* on (i.e., Lorentz invariant) **interval**

$$\Delta s^2 \equiv (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 \quad (7)$$

$$= (c\Delta t)^2 - (\Delta \ell)^2 \quad (8)$$

Note: interval can have $\Delta s^2 > 0, < 0, = 0$

Light pulse: $\Delta \ell = c\Delta t$

$\rightarrow \Delta s_{\text{light}} = 0$

∞ $\rightarrow$ light moves at $c$ in all frames!

Motion and time:

Consider two events, at rest in one frame:

$\Delta\vec{x}_{\text{rest}} = 0$ in rest frame, so

$\Delta s = c\Delta t_{\text{rest}}$: $c\times$ elapsed time in rest frame

In another inertial frame, relative speed $v$:

events separated in space by $\Delta x' = v\Delta t'$

$$\Delta s = \sqrt{c^2 \Delta t'^2 - \Delta x'^2} = \sqrt{c^2 - v^2}\,\Delta t' = \frac{1}{\gamma} c\Delta t' \qquad (9)$$

since $\Delta s$ same: infer $\Delta t' = \gamma\Delta t_{\text{rest}} > \Delta t_{\text{rest}}$

$\Rightarrow$ moving clocks appear to run slow

(special) relativistic time dilation

$\Rightarrow$ no absolute time (and no absolute space)



Note: more on Special Relativity in Director's Cut Extras to today's notes

H. Minkowski:

"Henceforth, space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality."

# The Speed of Massive Particles

Special relativity general rule: $v = p/E$
where $E$ is total energy (see Extras to notes)
good for particles of any mass $m \geq 0$
...and where we have and will set $c = 1$
you can show that with explicit $c$ factors, $v/c = cp/E$

but $E$ and $p$ also connected via
invariant $E^2 - p^2 = m^2$

$$v = \frac{\sqrt{E^2 - m^2}}{E} = \sqrt{1 - \left(\frac{m}{E}\right)^2} \tag{10}$$

*Q: implications? what if $m = 0$? $m \neq 0$?*

## Causality and Spacetime

any particle of total energy $E$, mass $m$

moves at speed $v(E) = \sqrt{1 - \left(\frac{m}{E}\right)^2}$

- massive particles have $0 \leq v < c$
- massless particles (e.g., $\gamma$) have $v = c$

$\Rightarrow$ $v = c = 1$ is universal speed limit

$\Rightarrow$ cannot transmit particles, info any faster

Geometrically:

at a given spacetime point $p$, light cone

**future light cone** at $p$

encloses region within which particles/info can move

i.e., region $p$ can influence

$\Rightarrow$ future light cone=causally connected to $p$

*past light cone* at $p$ *Q: ?*

**past light cone** at $p$

events in cone can send particles/info to $p$

i.e., region which could have influenced $p$

$\Rightarrow$ past light cone=causally connected to $p$

*Q: two events causally connected if?*

*Q: sufficient or just necessary?*

# What About Gravity?

A. Einstein (1905):
Newtonian dynamics $\longrightarrow$ relativistic dynamics
space, time $\longrightarrow$ spacetime forever more

Relativity and classical fields:

- E&M: Maxwell eqs relativistically *OK*! (*motivated* Lorentz, SR)
- Newtonian gravity: $\vec{g} = -\nabla\phi = -Gm/r^2\,\hat{r}$ and $\nabla^2\phi = 4\pi G\rho$ an *unmitigated disaster* *Q: Why?*

How to fix?
First attempt: analogy with electrostatics *Q: why?*

$$\nabla^2\phi - \partial_t^2\phi = 4\pi G\rho \tag{11}$$

- bad news: disagrees with expt (gives no light bending)
- good news: right “flavor”
  e.g., operator $\nabla^2 - \partial_t^2 \rightarrow$ waves $\rightarrow$ gravitational radiation!

# Mystic Pisa

**Experiment: Galileo** (Tower of Pisa?)
free fall independent of mass, size, shape, composition
*Q: lawyer's fine print?*

**Theory: Newton**
always: $\vec{F} = m\vec{a}$
gravity: mass is "coupling strength" $\Rightarrow \vec{F}_{\text{grav}} = m\vec{g}$
$\Rightarrow$ free fall has $\vec{a} = \vec{g} \rightarrow$ indep of object properties
interesting curiosity

**Theory: Einstein**
gravity is acceleration, so maybe acceleration is gravity
i.e., their physical effects indistinguishable/equivalent

## Equivalence Principle

T-shirt summary (R. Wald):
*all bodies fall the same way in a gravitational field*

an observer in free fall *Q: meaning?*
cannot perform any experiment to determine whether
she is in a gravitational field

an observer undergoing acceleration
cannot perform any experiment to determine whether
she is in a gravity field or an accelerating spacecraft

# Equivalence Principle and Spacetime

Gedankenexperiment: accelerating spaceship

- horizontal flashlight $\rightarrow$ drooping beam
- clocks at top & bottom

each flashes every $\tau_{\rm em} = 1$ sec $\rightarrow$ frequency $f_{\rm em} = 1/\tau_{\rm em}$
but asymmetry: top clock accel away from bottom flash
$\rightarrow$ relative speed of clocks changes during light transit
by amount $\delta v_{\rm top} \simeq -ah/c$ (receding from source)
$\rightarrow$ top observer sees freq Doppler shifted downward (redshift):

$$f_{\rm obs,top} \approx \left(1 + \frac{\delta v}{c}\right) f_{\rm em,bottom} \tag{12}$$

so top observer sees bottom flash interval as

$$\frac{\tau_{\rm obs} - \tau_{\rm em}}{\tau_{\rm em}} = \frac{\delta\tau}{\tau} = -\frac{\delta f}{f} \approx -\frac{\delta v}{c} = +\frac{ah}{c^2} \tag{13}$$

*Q: which means? and upon applying equivalence principle...?*

Equivalence Principle: in uniform gravity $g \to$ same results

- gravity bends light!

  `www: strong lensing`

- gravitational redshift/blueshift!
- gravitational time dilation

$$\frac{\delta t}{t} = \frac{\delta \lambda}{\lambda} \approx \frac{gh}{c^2} = \frac{\phi}{c^2} \tag{14}$$

attic clocks faster than basement clocks: verified experimentally!

`www: Pound-Rebka expt`

in weak gravity: shift $\approx \phi/c^2$

Note: gravity distorts

- light path (space)
- apparent frequency (time)

$\to$ gravity alters spacetime!

Einstein (1915): include gravity in spacetime

# Director's Cut Extras: Special Relativity

# Spacetime and Relativity

Pre-Relativity: space and time separate and independent
but *rotations* mix *space* coords, e.g.,

$$x' = x\cos\theta - y\sin\theta \quad ; \quad y' = y\cos\theta + x\sin\theta \tag{15}$$

without changing underlying vector (rotation of coords only)
transform rule holds not only for $\vec{x}$
but all other physical directed quantities: e.g., $\vec{v}, \vec{a}, \vec{p}, \vec{g}, \vec{E}$

Lesson: express & guarantee underlying rotational invariance
by writing physical law in vector form
e.g., $\vec{F} = m\vec{a}$ gives same physics for any coord rotation

In special relativity:
spatial rotations still allowed, but also...

"boosts" from one frame to another
with relative speed $\vec{v} = v\hat{x}$

$$t' = \gamma(t - vx/c^2) \quad (16)$$
$$x' = \gamma(x - vt) \quad (17)$$
$$y' = y \quad (18)$$
$$z' = z \quad (19)$$

- truly mix space and time $\rightarrow$ spacetime
- look like rotations, but 4-dimensional

$\rightarrow$ should express laws in terms of 4-D vectors:
"4-vectors," $t, x$ components transform via Lorentz

# Velocity, Momentum, Energy

Velocity:

for events separated by $dx^\mu = (dt, d\vec{x})$, put

$$u^\mu = \frac{dx^\mu}{ds} = (\frac{dt}{ds}, \frac{d\vec{x}}{ds}) \tag{20}$$

covariant: written this way, a 4-vector:

transforms in boost a la Lorentz

i.e., *components are different* in different frames

but underlying physical entity frame-independent

"like with space vectors and rotations"

norm ("length") of 4-velocity

$$u \cdot u = \left(\frac{dt}{ds}\right)^2 - \left(\frac{d\vec{x}}{ds}\right)^2 = \frac{dt^2 - d\vec{x}^2}{ds^2} = \frac{ds^2}{ds^2} = 1$$

same number for all observers: *invariant*

Now want 4-momentum $p^\mu$:
consider particle of (rest) mass $m$
where: rel. $p^i$ should go to $m\vec{v}$ for small $v$
try: $p^\mu = mcu^\mu$
space part: $\boxed{\vec{p} = \gamma m\vec{v}}$ rel momentum
time part:

$$p^0 = \gamma mc \approx \frac{1}{c}\left(mc^2 + \frac{1}{2}mv^2\right) = \frac{1}{c}\left(mc^2 + K\right) \tag{21}$$

can identify $\boxed{E_{\text{rel,tot}} = cp^0}$, but then
rest mass has energy $E_{\text{rest}} = mc^2$!

energy, momentum conservation $\rightarrow p^\mu$ cons
compact, unified treatment:
$(p^\mu)_{\text{init}} = (p^\mu)_{\text{fin}}$ (4 equations)

# The Charms of 4-Momentum

Invariant norm (everyone agrees)

$$p \cdot p = (p^0)^2 - (\vec{p})^2 = E^2 - \vec{p}^2 = m^2 \tag{22}$$

- rel. (total) energy is $E(p) = \sqrt{(cp)^2 + (mc^2)^2}$
- in rest frame: $\vec{p} = 0 \rightarrow E = mc^2$ "rest mass energy"
- define rel kinetic energy: $K_{\text{rel}} = E - mc^2$
  can show: $K_{\text{rel}} \rightarrow p^2/2m$ if $v \ll c$

Velocity
can show: $\vec{p}/E_{\text{tot}} = \vec{v}$
- non-rel: $Q$?

What if $m = 0$?
- $E^2 - \vec{p}^2 = 0 \rightarrow E = cp$: E is "all kinetic"
- $v = p/E = 1 = c$: moves at $c$ always!

# World Lines and Dynamics

for any observer (i.e., any coordinate system):
events along own worldline have

$$(\Delta s)^2 = (\text{observer's apparent elapsed time})^2 \qquad (23)$$

*Q: why?*

observers' total elapsed time going from events $A \rightarrow B$: $\Delta t = \int_a^b ds$
generically: in frame $x'$, elapsed time: $\Delta t = \int_a^b \sqrt{1 - v^2}\, dt'$

consider "race" from event $A$ to event $B$
accelerated vs non-accelerated ("free") observers
*Q: physical picture?*
can show: everyone agrees that
*non-accelerated* observer measures *longest* $\Delta t$
*Q: this is huge–why? what's special about such observers in SR?*

non-accelerated observer $\rightarrow$ no forces
i.e,. a free body!

so in Special Relativity:
of all trajectories from events $A \rightarrow B$
free bodies have max $\int ds$

but free body trajectory is natural motion!

Implications
$\Rightarrow$ free body follows extremum of $\int ds$
law of motion!
i.e., variation $\delta \int ds = 0$ selects physical worldline!
$\Rightarrow$ twin "paradox" is not *Q: why?*