Astro 596/496 PC
Lecture 9
Feb. 8, 2010

Announcements:
- PF2 due next Friday noon
- High-Energy Seminar right after class, Loomis 464:
  Dan Bauer (Fermilab)
  "Recent Results from the Cryogenic Dark Matter Search"

Last time:
- spacetime–Aristotle to Galileo/Newton to Einstein
- equivalence principle *Q: namely?*
- rocket gedankenexperiment
  Q: implications for light trajectory?
  Q: implications for photon frequency/wavelength/energy?
  Q: implications for clocks?

Equivalence Principle: in uniform gravity g

$\rightarrow$ same results as rocket accelering with $a = g$

- gravity bends light!

  `www: strong lensing`

- gravitational redshift/blueshift!

over light travel time $t \approx \delta h/c$, $\delta v_{\rm obs} \approx at \approx g\delta h/c$

$$\frac{\delta\lambda}{\lambda} \approx \frac{\delta v_{\rm obs}}{c} \approx \frac{g\,\delta h}{c^2} = \frac{\delta\phi}{c^2} \tag{1}$$

- gravitational time dilation:

  light as clock, tick interval $\tau = 1/f = \lambda/c$

$$\frac{\delta\tau}{\tau} = \frac{\delta\lambda}{\lambda} \approx \frac{g\delta h}{c^2} = \frac{\delta\phi}{c^2} \tag{2}$$

Note: gravity distorts

- light path (space)
- apparent frequency (time)

$\rightarrow$ gravity alters spacetime!

Einstein (1915): include gravity in spacetime

General Relativity

Newton (1687): Universal Gravitation
gravity is a force (field) that couples to mass
▷ matter tells gravity how to force
▷ gravity force tells matter how to move

Einstein (1915): General Relativity
gravity is spacetime curvature: not a force!
⋆ matter tells space how to curve
⋆ space tells matter how to move

Curved Spacetime?

Curved space: geometric constructions in space
(triangles, rectangles, circles... *Q: how define?*)
give non-Euclidean results *Q: namely?*

Q: so–curved spacetime?

Spacetime Curvature

Test: (Feynman Lectures II, Chapter 42)

- construct geometric object in spacetime
- are properties Euclidean?

Case 1: Minkowski Space (i.e., special relativity, no accel)
(1-D) interval ("line element") for events separated by (dt, dx)

$$ds^2 = dt^2 - dx^2 \tag{3}$$

Construct rhombus: two observers go from events A to B

▷ obs 1: go left at $v = 0.5c$ for 10 s, then wait 10 s
▷ obs 2: wait 10 s, then go left at $v = 0.5c$ for 10 s

Q: spacetime diagram?

result is Euclidean *Q: why?*

⇒ Minkowski spacetime is **not curved = flat**

Case 2: Surface of Earth (i.e., const accel: gravity)

(1-D) line element:

$$ds^2 = \left(1 + \frac{2\phi}{c^2}\right) dt^2 - \left(1 + \frac{2\phi}{c^2}\right)^{-1} dx^2 \quad (4)$$

where $\phi = \phi(x)$: time-independent

Construct rhombus: two observers go from events A to B

▷ obs 1: go up 1 km, then wait 10 s

▷ obs 2: wait 10 s, then go up 1 km

Q: spacetime diagram?

result is **not** Euclidean *Q: why?*

⇒ Earth's spacetime is **curved**!

gravity ⇔ spacetime curvature

GR on a T-Shirt

General Relativity spirit and approach:
like special relativity, only moreso

Special Relativity concepts retained:

- **spacetime**: events, relationships among them
- **interval** gives observer-independent (invariant) measure of "distance" between events
- Special Relativity is a special case of GR
  SR: no gravity → no curvature → "flat spacetime"
  GR limit: gravity sources→0 give spacetime→Minkowski

GR: Special Relativity concepts generalized

- gravity encoded in spacetime structure
- spacetime can be curved
- coordinates have no intrinsic meaning

The Metric

Fundamental object in GR: **metric**
consider two nearby events, separated by
coordinate differences $dx^\mu = (dx^0, dx^1, dx^2, dx^3)$
GR (in orthogonal spacetimes) sez:
interval between them given by **"line element"**

$$
\begin{aligned}
ds^2 &= A(x)\,(dx^0)^2 - B(x)\,(dx^1)^2 - C(x)\,(dx^2)^2 - D(x)\,(dx^3)^2 \\
&\equiv \sum_{\mu\nu} g_{\mu\nu} dx^\mu dx^\nu \equiv g_{\mu\nu} dx^\mu dx^\nu
\end{aligned}
$$

where the **metric tensor** $g_{\mu\nu}$

- in this case (orthogonal spacetime): $g = \text{diag}(A, B, C, D)$
- components generally are functions of space & *time* coords
- is symmetric, i.e., $g_{\mu\nu} = g_{\nu\mu}$
- encodes all physics (like wavefunction in QM)

Q: if no gravity=Minkowski, what's the metric?

physical interpretation of interval: like in SR

$$ds^2 = (\text{apparent elapsed time})^2 - (\text{apparent spatial separation})^2$$

⋆ observers have *timelike* worldlines: $ds^2 > 0$

⋆ light has *null* $ds = 0$ worldlines

Simplest example: Minkowski space (Special Relativity)

$g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$: constant values proper spatial distances:

- i.e., results using meter sticks
- measured simultaneously ($dx^0 = 0$)

length element:

$$d\ell^2 = -ds^2 = d\ell_1^2 + d\ell_2^2 + d\ell_3^2 = g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + g_{33}(dx^3)^2$$

space (3-)volume element:

$$\begin{aligned} dV_3 &= d\ell_1 d\ell_2 d\ell_3 \\ &= \sqrt{|g_{11}g_{22}g_{33}|}\; dx^1 dx^2 dx^3 \end{aligned}$$

spacetime 4-volume element:

$$\begin{aligned} dV_4 &= d\ell_0 dV_3 = \sqrt{|g_{00}g_{11}g_{22}g_{33}|}\; dx^0 dx^1 dx^2 dx^3 \\ &= \sqrt{|\det g|}\; dx^0 dx^1 dx^2 dx^3 \end{aligned}$$

Example: Minkowski space, Cartesian coords

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2$$

length: $d\ell^2 = dx^2 + dy^2 + dz^2$

3-volume: $dV_3 = dx\, dy\, dz$

4-volume: $dV_4 = dx\, dy\, dz\, dt$

Example: Minkowski space, spherical coords

$$ds^2 = dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2$$

length: $d\ell^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$

3-volume: $dV_3 = r^2 \sin\theta\, dr d\theta d\phi \equiv r^2 dr d\Omega$

4-volume: $dV_4 = r^2 dr d\Omega dt$

Relativistic Cosmology

Cosmological Spacetimes

Want to describe spacetime of the universe
to zeroth order: homogeneous, isotropic

1. at each spacetime point
   at most *one* observer sees isotropy
   call these fundamental observers
   roughly: "galaxies" i.e., us
   (strictly speaking, we don't qualify) *Q: why?*

2. isotropy at each point $\longrightarrow$ homogeneity
   but can be homogeneous & not isotropic

3. homogeneity and isotropy $\longrightarrow$ symmetries

U. is **“maximally symmetric”**

$\longrightarrow$ greatly constrain allowed spacetimes

i.e., allowed metrics

The Cosmic Line Element

cosmological principle:
can divide spacetime into time "slices"
i.e., 3-D spatial (hyper) surfaces

$\triangleright$ populated by fundamental observers

$\triangleright$ with coords, e.g., (t, x, y, z)

$\triangleright$ choose FO's to have $d\vec{x} = 0$
i.e., spatial coords are **comoving**

on surface: fundamental observers must all have
$ds^2 = dt^2 \rightarrow$ i.e., $g_{tt} = const = 1$ *Q: why?*
$\rightarrow g_{tt}$ indep of space, time

these give:

$$ds^2 = dt^2 - g_{ii}(dx^i)^2 \quad (5)$$

Cosmological Principle and the Cosmic Metric

homogeneity and time

no space dependence on $d\ell_0 = dt$

- can define cosmic time t (FO clocks)
- at fixed t, time lapse dt not "warped" across space

homogeneity and space

- at any t, properties invariant under translations
- no center
- can pick arbitrary point to be origin
- e.g., here!