Astronomy 210
Homework Set #10: Simulation Option
Due in class: Friday, April 22
Total Points: 50 + 10 bonus

Choosing a Computational Project

This week's homework uses computer-based analysis to get important astrophysics results. There are *two* options, this one involving basic programming to simulate a star, the other involving collection and analysis to derive properties of galaxies in the local and distant universe.

You should choose *one* of these projects to do.

This Option: A White Dwarf in Your Computer

Use a computer to solve the equations of stellar structure for a white dwarf!

While the writeup for this project may seem long, mostly that's because I've already done the necessary derivations for you. You only need to answer the questions labeled with a), b) and so forth. You will also have to write a small computer code, but if you want you can follow the outline in Part III very closely, which should make it very easy.

Part I: The Lane-Emden Equation

We'll be interested in "cold" stars, for which we only have to solve for hydrostatic equilibrium. In class, we saw how the idea of hydrostatic equilibrium applied to the bulk of the *whole star* can be used to relate the *average* pressure in a star to it's mass and radius. In fact, hydrostatic equilibrium also applies to *each point* in the star, and relates the pressure (in fact, the pressure gradient) at each point to the gravity and density at that point. This is done via the equation of hydrostatic equilibrium

$$\frac{dP}{dr} = -\rho \frac{Gm}{r^2}, \tag{1}$$

which is derived on p. 311 of your text, which also derives the mass equation

$$\frac{dm}{dr} = 4\pi r^2 \rho \tag{2}$$

These equations, together with an equation of state relating P to ρ, are enough to specify the structure of the star.

It turns out that for white dwarfs, the equation of state takes the particularly simple form

$$P = K\rho^{1+1/n}. \tag{3}$$

This is called a polytropic equation of state. K is a constant, and n is the polytropic index.[1] K and n are different for different kind of objects and different physical regimes,

[1] In class, we wrote this equation of state in another common form, $P = K\rho^{\gamma}$. In this case, K is the same as in eq. (3), while $\gamma = 1 + 1/n$, or $n = 1/(\gamma - 1)$.

but in part V we'll focus on high density white dwarfs, for which K and n are known.

In order to bring these equations into a more convenient form, we can solve equation (1) for m, take a derivative, and insert into (2), which yields

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho. \tag{4}$$

We now substitute (3) into (4), and find

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{Kr^2}{\rho}\frac{d\rho^{1+1/n}}{dr}\right) = -4\pi G\rho, \tag{5}$$

which can be rewritten

$$\frac{1}{r^2}\frac{d}{dr}\left(Kr^2(n+1)\frac{d\rho^{1/n}}{dr}\right) = -4\pi G\rho. \tag{6}$$

At this point it turns out to be very convenient to make the substitutions

$$\rho = \rho_c\theta^n \tag{7}$$

and

$$r = a\xi, \tag{8}$$

where ρ_c is the density at the center, and a is

$$a = \left(\frac{(n+1)K\rho_c^{(1/n)-1}}{4\pi G}\right)^{1/2}. \tag{9}$$

θ and ξ are now dimensionless – all the physical dimensions have been absorbed in ρ_c and a. Equation (6) now takes the easy form

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n. \tag{10}$$

This is the famous *Lane-Emden* equation. This is a second order, ordinary differential equation for the "density profile" θ, as a function of the "radius" ξ. (It is "second order", because it involves up to second derivatives of θ). We also have to determine boundary conditions at $\xi = 0$ (i.e. $r = 0$). Obviously, at the center the density has to be the central density, and hence

$$\theta = 1 \qquad \text{at} \qquad \xi = 0. \tag{11}$$

Also, the derivative of the density has to vanish at the center, and therefore

$$\frac{d\theta}{d\xi} = 0 \qquad \text{at} \qquad \xi = 0. \tag{12}$$

These are the two boundary conditions that are necessary for a second order equation.

Our goal is to find a function $\theta(\xi)$ that satisfies the Lane-Emden equation (10). In general, this has to be done numerically, as in part III. Once we have found this function, the radius of the star is determined by the point at which θ goes to zero. We will call this point ξ_1:

$$\theta(\xi_1) = 0. \tag{13}$$

Once we know ξ_1, the radius of the star is obviously

$$R = a\xi_1 = \left(\frac{(n+1)K\rho_c^{(1/n)-1}}{4\pi G} \right)^{1/2} \xi_1. \tag{14}$$

We can also find the mass

$$\begin{aligned} M &= m(R) = 4\pi \int_0^R \rho r^2 dr = 4\pi a^3 \rho_c \int_0^{\xi_1} \xi^2 \theta^n d\xi & (15) \\ &= -4\pi a^3 \rho_c \int_0^{\xi_1} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) d\xi = 4\pi a^3 \rho_c \xi_1^2 |\theta'(\xi_1)|, & (16) \end{aligned}$$

where we have inserted (10) for θ^n, and where θ' is short for $d\theta/d\xi$. Inserting (9) for a, we finally find

$$M = 4\pi \left(\frac{(n+1)K}{4\pi G} \right)^{3/2} \rho_c^{(3-n)/(2n)} \xi_1^2 |\theta'(\xi_1)|. \tag{17}$$

Question 1 Show that for $n = 3$, the mass is independent of the central density.

Part II: An Analytical Solution

For $n = 1$, an analytical solution to the Lane-Emden equation (10) can be found:

$$\theta = \frac{\sin(\xi)}{\xi}. \tag{18}$$

Question 2 Show that (18) satisfies the Lane-Emden equation (10). (Insert the analytical solution into the left hand side, and show that you get the right hand side!)

Question 3 What is ξ_1 for $n = 1$?

The solution (18) also satisfies the boundary conditions. We will use this solution as a check for our computer program.

Part III: Numerical Implementation

We now want to integrate the Lane-Emden equation (10) on a computer! Instead of integrating it in the second order form, however, it is easier to make two first order equations out of it. One way to do that is to introduce a new variable

$$\phi = \xi^2 \frac{d\theta}{d\xi}. \tag{19}$$

Inverting this equation we find a first order equation for θ

$$\frac{d\theta}{d\xi} = \frac{\phi}{\xi^2}, \tag{20}$$

and inserting (19) into (10) now yields a first order equation for ϕ:

$$\frac{d\phi}{d\xi} = -\xi^2\theta^n. \tag{21}$$

The last two equations can now be integrated very easily with a computer. If you have used numerical libraries before you can do that and use as fancy a routine as you want. If not, you can do something very simple, as for example the following. First, let's replace the d's by Δ's:

$$\Delta\theta = \frac{\phi}{\xi^2}\Delta\xi \tag{22}$$

$$\Delta\phi = -\xi^2\theta^n\Delta\xi. \tag{23}$$

Now start at $\xi = 0$, with $\theta = 1$ and $\phi = 0$. Choose a fairly small $\Delta\xi$ – for example, $\Delta\xi = 10^{-3}$ may do the job. $\Delta\xi$ controls the accuracy of your calculation, and you should play with it to see its effect. Integrate from ξ to $\xi + \Delta\xi$ by adding $\Delta\theta$ to θ and $\Delta\phi$ to ϕ. Repeat this procedure until θ becomes negative – and you're done!

Note that there is one tricky part, and that is that at $\xi = 0$ you cannot divide by ξ in order to find $\Delta\theta$. However, ϕ is zero at $\xi = 0$ also, and it turns out that both right hand sides vanish at $\xi = 0$. I therefore coded up a little `if` statement, which sets the right hand sides to zero if $\xi = 0$, and evaluates them according to equations (22) and (23) otherwise.

Your whole code may be very short and may look like something like this:

```
double n = 1.0;
double dxi = 0.001;
double theta = 1.0;
double phi = 0.0;
double xi = 0.0;

do {
  if (xi == 0.0) {
    phi = phi + 0.0;
    theta = theta + 0.0;
  } else {
    theta = theta + dxi * (phi / (xi*xi));
    phi = phi - dxi * (xi*xi * pow(theta,n) );
  }
  xi = xi + dxi;
} while (theta > 0.0);
```

This is a `c++` code, but you can use any programming language that you like. You'll have to add a few lines for output! This is probably the most primitive code you'd come up with – anything more fancy is welcome, too!

Question 4 Write a numerical code to solve the Lane-Emden equation. Print out the code and attach it to your report.

Part IV: Code Check

Question 5 Run your code for $n = 1$. What do you find for ξ_1? Compare with your result from II.b)! What happens if you increase/decrease $\Delta\xi$? Explain.

Question 6 Make a plot of the density in units of the central density (i.e., plot θ^n) versus ξ. In one graph, plot both your numerical result for θ and the analytical solution (18). (Note: for $n = 1$, we obviously have $\theta^n = \theta$, but the difference between θ and ρ will be important below!) Also make a separate plot the "residual" of your calculation–i.e., plot the difference $\delta\theta = \theta_{\text{code}} - \theta_{\text{analytical}}$ versus ξ, and/or plot the percent error $100\delta\theta/\theta_{\text{analytical}}$ versus ξ. If your code were infinitely accurate, it would give $\delta\theta = 0$. Thus, the size of $\delta\theta$ and of the percent error gives you a measure of the accuracy of your code. How good is your result? How does the size of the residual depend on your choice of stepsize $\Delta\xi$?

Part V: White Dwarfs

For white dwarfs with high densities, the equation of state is well approximated by a polytropic equation of state with polytropic index $n = 3$. The constant K in the polytropic equation of state then is

$$K = \frac{3^{1/3}\pi^{2/3}}{2^{4/3}4}\frac{\hbar c}{m_p^{4/3}}, \tag{24}$$

where m_p is the proton mass.

Question 7 Use your code to integrate the Lane-Emden equation for $n = 3$. What is ξ_1? What is $|\theta'(\xi_1)|$?

Question 8 Plot the density in units of the central density (i.e. θ^n) as a function of ξ. Compare with your plot for $n = 1$. What do people mean when they say that $n = 3$ polytropes are "centrally condensed"?

Question 9 What is the mass of a high density white dwarf? If you have done everything right, you will have rediscovered the Chandrasekhar Mass!

A Final Note: For low density white dwarfs the polytropic index is $n = 3/2$, which is the value quoted in the textbook. For intermediate values, the polytropic index takes an intermediate value also. Such white dwarfs have masses a little less than the Chandrasekhar Mass. (For example, Sirius B, which we talked about in class, has a mass of about $1.05M_\odot$.)

Bonus [10 points]

Come up with your own original mnemonic for the stellar spectral types OBAFGKM, to replace the standard "O Be A Fine Girl/Guy Kiss Me." Submissions should be no worse than PG-13 rated to receive credit. For style points, extend this to include the new spectral types L and T (i.e., create something that can also work for OBAFGKML or even OBAFGKMLT). Prizeworthy entries will be shown in class.

To receive credit, submit your entry on Compass by the start of class on Friday April 22.