Astronomy 210 Fall 2010
Homework Set #3

Due in class: Friday, Feb. 11 Total Points: 50 + 5 bonus

1. *Alternative Universes: Kepler's Laws.* [**5 points**] Consider an alternative universe, in which all of the laws of physics are the same as ours, *except gravity.* In this alternative universe, the strength of the gravitational force is still proportional to the masses of the two gravitating objects, and is still attractive and radial, but now depends differently on distance r. So we will write the Alternative gravitational force law as $\vec{F} = -m_1 m_2 f(r)\hat{r}$, with $\hat{\mathbf{r}} = \mathbf{r}/|\mathbf{r}|$, and $f(r)$ an unknown (for the moment) function of r.

   In this strange universe, Alternative Kepler finds that for planets in circular orbits $r = a = const$, the orbit radius a and period P are related by the "Alternative Kepler's Third Law" of

   $$P \propto a^2 \tag{1}$$

   Using this information, show how the function $f(r)$ in the force law depends on r. Comment on the way that Kepler's third law encodes information about the gravitational force law.

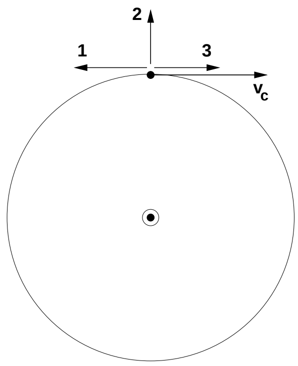
2. *Strong Gravity and Light.* Consider an object of mass M and radius R.

   (a) [**5 points**] If we keep the mass fixed but compress the object, find the radius at which the escape speed from the object's surface is equal to the speed of light c (which is always constant); call this radius R_S.

   (b) [**5 points**] Imagine an object of mass M and radius $R \leq R_S$. How would this object look to a distant observer?

   (c) [**5 points**] Calculate R_S for an object with the mass of the Sun $M_\odot = 2.0 \times 10^{30}$ kg; note that $c = 3.0 \times 10^8$ m/s. Compare your answer to the actual solar radius $R_\odot = 7.0 \times 10^8$ m, and comment.

3. *Cosmic trash disposal.* Consider an object ("cosmic trash") of mass m which is in a circular orbit around the sun with speed v_c and radius a. To completely dispose of the waste, the options are either to drop it into the sun, or launch it out of the solar system.



   (a) [**5 points**] To drop the trash into the sun requires that we remove the object's angular momentum, which costs us energy. Namely, we want to boost the material (i.e., change the velocity by firing rockets) so that it has a velocity $v = 0$ with respect to the sun; it then falls into the sun. To do this, the material must be boosted by a velocity $v_b = v_c$ in the opposite direction its original orbit direction (vector 1 in the figure). Compute the boost energy required (in terms of m and v_c), where the boost is assumed to be instantaneous and thus all happens at distance $r = a$ from the sun.

   (b) [**5 points**] To fling the material from the solar system also requires energy, as we want the object to be just unbound: $KE + PE = 0$. We can do this by boosting the object so that $KE = -PE = GM_\odot m/a$. If the boost (speed v_b) is aimed away from the sun (vector 2), this is perpendicular to the object's initial speed, and so the new speed with respect to the sun is $v^2 = v_b^2 + v_c^2$. Compute the speed v_b needed to satisfy $KE + PE = 0$ in this case, and the boost energy needed (in terms of m and v_c). How does this compare with the energy in part (a)?

(c) **[5 points]** Finally, we can eject the material from the solar system as in part (b), but now boost the material with speed v_b in the same direction as its initial orbit (vector 3). Then the object's new speed with respect to the sun is $v = v_b + v_c$. Compute the speed v_b needed to satisfy $KE + PE = 0$ in this case, and the boost energy needed (in terms of m and v_c). How does this compare with the energies in parts (a) and (b)? Which of these is the cheapest means of astronomical waste disposal?

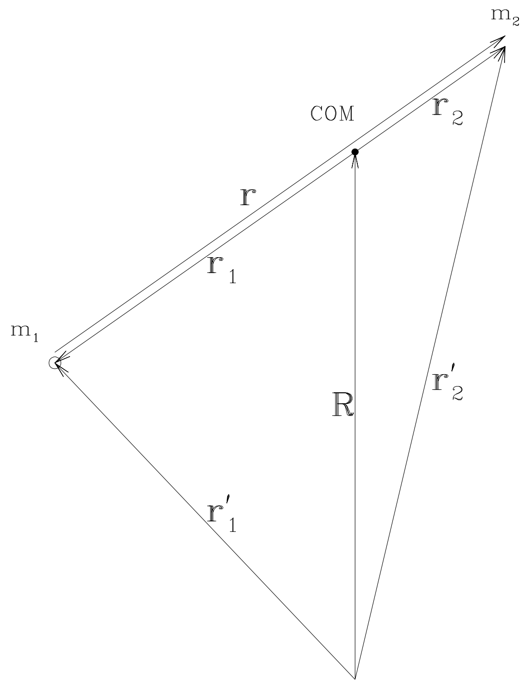
4. *Center of Mass.* Consider two bodies with masses m_1 and m_2, and position vectors $\mathbf{r_1'}$ and $\mathbf{r_2'}$ relative to some arbitrary origin; see figure at right. Remember that the vector to the center of mass (with respect to this same origin) is

$$\mathbf{R} = \frac{m_1\mathbf{r_1'} + m_2\mathbf{r_2'}}{m_1 + m_2} \tag{2}$$

and that the vector connecting the two bodies is $\mathbf{r} = \mathbf{r_2'} - \mathbf{r_1'}$. Also recall that the reduced mass is

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \quad .$$

Assume that the only force acting on the masses is the gravitational attraction between them. *Hint*: you life will be easier if you treat the vectors as objects rather than break them down into components. For example, while one can write $d/dt\ (x, y, z) = (dx/dt, dy/dt, dz/dt) = (\dot{x}, \dot{y}, \dot{z})$, you can also just use $\vec{r} = (x, y, z)$ and put $d/dt\ \vec{r} = \dot{\vec{r}} = \vec{v}$, and $d/dt\ \vec{v} = \ddot{\vec{r}}$. Also note that in this notation, $r = |\vec{r}| = \sqrt{\vec{r}\cdot\vec{r}}$.



(a) **[5 points]** By differentiation of eq. (2), show that $\ddot{\mathbf{R}} = 0$. What does this mean physically about the motion of the system?

(b) **[5 bonus points]** Show that when we pick the center of mass as the origin, as in the figure above, the particle displacements are

$$\mathbf{r_1} = -\frac{m_2}{m_1 + m_2}\mathbf{r} \quad \text{and} \quad \mathbf{r_2} = \frac{m_1}{m_1 + m_2}\mathbf{r}$$

(c) **[5 points]** Using the result from (b) Show that $r_1/r_2 = m_2/m_1$, where $r_1 = |\mathbf{r_1}|$ and $r_2 = |\mathbf{r_2}|$.

(d) **[5 points]** Finally, consider the case of circular orbits. In Kepler's laws, the radius $a = r_1 + r_2$. Using this, and part (c), calculate the Sun's distance to the Earth-Sun center of mass. How does this compare to the size of the Sun?