Astro 210

Lecture 9

Feb 7, 2011

Announcements

- HW3 due Friday
  HW1 Q8 bonus still available
- iClicker scores to date are posted
- Planetarium: new shows added
  registration, report form, info online

Homework, Collaboration, and Academic Honesty

From the course Syllabus, emphasis added:

> Discussing course material with your classmates is in general not only allowed but in fact a good idea. However, each student is expected to do *his or her own work*. On homework, you may discuss the questions and issues behind them, but *you are responsible for your own answers*.

It should be obvious but:

you are *not* to consult homework solutions from previous semesters

Last Time: Sir Isaac Weighs In

Newton's Laws of motion
I. inertia *Q: just a special case of Newton II?*
II. $\vec{F} = m\vec{a}$ *Q: fortunetelling & archæology?*
III. action-reaction

Q: when/where/to what do Newton's laws of motion apply?

Newtonian Gravitation
Q: magnitude of gravity force between masses m and M with distance r?
Q: direction of the force?

Kepler from Newton

Kepler I: Orbits are ellipses

Newton: bound orbits due to gravity are ellipses: check!

Kepler II: Equal areas in equal times

Newton: consider small time interval dt

move angle $d\theta = \omega dt$

sweep area

diagram: top view: path, $d\theta, \vec{r}, \vec{v}, \vec{v}_\theta$

$$dA = \frac{1}{2}r^2 d\theta = \frac{1}{2}r^2 \omega dt \tag{1}$$

but $\omega = v_\theta / r$, where $\vec{v_\theta} \perp \vec{r}$

$\Rightarrow$ swept area

$$dA = \frac{1}{2}r^2 \frac{v_\theta}{r} dt = \frac{1}{2} r v_\theta dt \tag{2}$$

$\Rightarrow$ swept area

$$dA = \frac{1}{2} r^2 \frac{v_\theta}{r} dt = \frac{1}{2} r v_\theta dt \quad (3)$$

finally, $r v_\theta = |\vec{r} \times \vec{v}| = |\vec{L}|/m$
Q: why?, so

$$dA = \frac{1}{2} \frac{L}{m} dt \quad (4)$$

Woo hoo! were' home free! *Q: why?*

But $L = const$ for radial force ($\vec{r} \times \vec{F} = 0$)
so

$$\frac{dA}{dt} = \frac{L}{2m} = const \tag{5}$$

Kepler II! $\rightarrow$ comes from ang. mom. cons.!

iClicker Poll: Orbits and Angular Momentum

Consider two objects at 1 AU:

- a planet in a circular orbit
- a spacecraft moving directly towards the Sun

Which has the larger *magnitude* of angular momentum?

A the planet

B the spacecraft

C either, depending on the speed of the spacecraft

Kepler III: $a^3 = kP^2$

Newton: can prove generally for elliptical orbits

bad news: price is lotsa algebra

good news: simple to do for circular orbits

circular $\rightarrow r = a$, and $v^2 = GM/a$

but also $v = 2\pi a/P$ *Q: why?*

$$
\begin{aligned}
v^2 = \left(\frac{2\pi a}{P}\right)^2 &= \frac{4\pi^2 a^2}{P^2} & (6) \\
&= \frac{GM}{a} & (7) \\
\Rightarrow a^3 &= \left(\frac{GM}{4\pi^2}\right) P^2 & (8)
\end{aligned}
$$

check!

bonus: $k = GM/4\pi^2$ depends on mass of central object

$\rightarrow$ same k for all planets

Energy

For "test" particle m moving due to gravity of M

Gravitational potential energy: *Q: why "potential"?*

$PE = -GMm/r$

Kinetic energy:

$$KE = \frac{1}{2}m\vec{v}^2 = \frac{1}{2}m(v_x^2 + v_y^2 + v_z^2) = \frac{1}{2}m\dot{\vec{r}}^2 \qquad (9)$$

Total energy:

$TE = KE + PE = -GMm/r + \frac{1}{2}m\dot{\vec{r}}^2$

key result: $d(TE)/dt = 0$

$\rightarrow$ total energy conserved!

that is: value of TE the same for all time!

Orbits Revisited

Bound orbits (circle & ellipse): in polar coordinates

$$r(\theta) = \frac{(1-e^2)a}{1+e\cos\theta} \tag{10}$$

Circle radius $r = a = const$, eccentricity $e = 0$
recall: circular orbit has constant speed $v_c^2 = GM/r$

$$\begin{aligned} PE &= -\frac{GMm}{r} \quad < 0 && (11) \\ KE &= \frac{1}{2}mv_c^2 = \frac{1}{2}m\frac{GM}{r} = \frac{1}{2}\frac{GMm}{r} = -\frac{1}{2}PE && (12) \\ \Rightarrow TE &= KE + PE = PE/2 = -|PE|/2 < 0 && (13) \end{aligned}$$

$TE < 0$: negative? yes!
Q: what does it mean to have negative energy?

for orbiting system $TE < 0$:
$\rightarrow$ have to *supply* energy to system to break it apart

Why? when particles are *at rest* and "very" *far apart*
$KE = mv^2/2 = 0$
$PE = GMm/r \rightarrow 0$ *Q: how far apart is this?*
and so $TE = KE + PE = 0$: zero total energy
But if start in closed orbits (circular or elliptical): $TE < 0$
$\rightarrow$ To "break" the system from closed orbits, must *add* energy
But energy is conserved $\rightarrow$ not spontaneously added
so system is **bound**
$\Rightarrow$ can't fall apart without external influence

Note: $KE = -PE/2 = |PE|/2$ generally true for
gravitating systems in equilibrium:
"virial theorem"

ellipse: semimajor axis a, eccentricity $0 < e < 1$

turns out: TE depends only on a, not e

from cons of energy

$TE = -GMm/r + \frac{1}{2}mv^2 = -GMm/2a < 0 \rightarrow$ bound

can show

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right) \tag{14}$$

"vis viva" equation ("life force")

discovered prior to concept of energy

handy: gives total speed v at any radius r

Q: at which r is $v = 0$? how does this work for a circular orbit?

Q: for a given orbit (fixed e), when is v max?

Unbound Orbits

Note that both parabolic and hyperbolic orbits are not periodic – do not close on themselves "one-way ticket" past the central object

Parabola

$e = 1$

$$r = \frac{2p}{1 + \cos\theta} \tag{15}$$

p is distance of closest approach

for parabolic orbit:

$TE = 0$ exactly! $\rightarrow KE = -PE$ exactly! very special case!

$\Rightarrow GM/r = \frac{1}{2}v^2$

So at $r = \infty$, $v = 0$

to have this orbit, launch from r with speed

$v_{\text{launch}} = \sqrt{2GM/r}$

iClicker Poll: Orbits

given: test particle m, at distance r from gravitating body M
for test particle to have total energy $TE = 0$
launch from r with speed $v_0 = \sqrt{2GM/r}$

Q: what happens if launch with speed $v > v_0$?

A particle will be in a bound orbit: circle or ellipse

B particle will be unbound, with speed $v \to 0$ as $r \to \infty$

C particle will be unbound, with speed $v > 0$ as $r \to \infty$

Q: why is v_0 a special speed?

Escape Speed

At radius r, define **escape speed** $v_{\text{esc}} = \sqrt{2GM/r}$

- if launch from r with $v_{\text{launch}} < v_{\text{esc}}$
  then $TE < 0$: fall back! (elliptical orbit)
- if launch from r with $v_{\text{launch}} > v_{\text{esc}}$
  then $TE > 0$: escape "easily": $v > 0$ at $r = \infty$
- if launch from r with $v_{\text{launch}} = v_{\text{esc}}$ exactly
  then $TE = 0$ exactly, "just barely" escape

So: escape speed is *minimum speed* needed to leave a gravitating source

Example: escape speed from earth
$v_{\text{esc}} = 11$ km/s = 25,000 mph!
predict the future: if toss object with $v < 25,000$ mph, falls back
but if $v > 25,000$ mph *Q: example?* never returns!

finally, the more "generic" unbound orbit:

hyperbola

$$r(\theta) = \frac{(e^2 - 1)a}{1 + e\cos\theta} \tag{16}$$

$e > 1$, $TE > 0$

$v > 0$ at $r = \infty$: nonzero speed far from M

Recall: at large r, hyperbola $\rightarrow$ *straight line*

But Newton says: $d\vec{v}/dt = -GM/r^2\ \hat{r}$

so as $r \rightarrow \infty$, then $d\vec{v}/dt \rightarrow 0$

$\Rightarrow$ gravity negligible, $\vec{v} \rightarrow$ const: *free body*=straight line!

orbit of unbound "flyby":

distant nearly free body $\rightarrow$ passing: pulled toward M

$\rightarrow$ distant deflected nearly free body