Astronomy 501 Spring 2013
Problem Set #1

Updated Jan 23
Due in class: Friday, Jan. 25
Total points: 7+1

Note: Your homework solutions should be legible and include all calculations, diagrams, and explanations. The TA is not responsible for deciphering unreadable or illegible problem sets! Also, homework is graded on the method of solution, not just the final answer; you may not get any credit if you just state the final answer!

You may discuss with other students, but you are responsible for your own answers: you must understand your solutions, and you must write them yourself *in your own words.*

1. *Radiation transfer of your room.* Radiation transfer begins at home! Model the room you are in as a non-absorbing interior, with blackbody walls and surfaces at all the same room temperature T.

   (a) [**0.5 points**] Solve the transfer equation with appropriate boundary conditions to show that everywhere inside the room, the specific intensity $I_\nu = B_\nu(T)$.

   (b) [**0.5 points**] Find an expression for the frequency-integrated number density n of thermal photons inside the room.

   Evaluate your expression numerically, using a realistic value for T (in Kelvin!). Express your answer in units of photons/cm^3.

   Also estimate the number density of gas molecules in the room (mostly nitrogen N_2 molecules), in units of molecules/cm^3. Which particles are more abundant inside the room: matter or radiation?

2. *Olber's Paradox.* Prior to Edwin Hubble's work in the 1920's enlarging the cosmic distance scale, it was often implicitly assumed that the universe was static, infinitely large, infinitely old, and filled with (unchangingly luminous) stars; let's call this the "naïve cosmology." However, J. de Cheseaux in 1744, and more famously Heinrich Olbers in 1826, noticed that this seemingly straightforward extrapolation of the observed celestial sphere leads to predictions so grossly incorrect that any naked-eye glance the night sky can rule them out.

   We wish to find the brightness of the night sky in the naïve cosmology. For simplicity, let all stars be identical, with radius $R_\odot$, and emit blackbody radiation at a surface temperature $T_\odot$. We will consider a case in which there is a uniform (homogeneous) spherical distribution of stars, with a number density $n_\star$ and radius r. In the naïve cosmology, the universe is infinite is size, so this would correspond to the case $r \to \infty$.

   Note that in this naïve universe (but not in ours!) we ignore expansion, redshifting, and time dilation, and assume that space has the usual Euclidean geometry.

   (a) [**1 points**] Find the starlight emission coefficient j_ν for the naïve cosmology. *Hint:* you will want to find and expression for the specific luminosity L_ν of each star, in terms of the variables you have been given.

(b) [**0.5 points**] Write down and solve the equation of radiation transfer for this situation, ignoring for now that the stars will absorb light incident on them. Express your answer in terms of the radius r. You should find an (isotropic) sky intensity for the naïve universe $I_\nu \to \infty$. Interpret your result; what is the physical reason for your very unphysical answer?

(c) [**1 point**] Your answer for part (b) is too simple even for the naïve cosmology, because the stars themselves can absorb light. Now allow stars to absorb light, with a cross section at all wavelengths equal to the geometric cross section of the star. Find the absorption coefficient α_ν and the source function S_ν.

Write the equation of radiation transfer for this situation, find the solution for $I_\nu(r)$ and for the integrated intensity $I(r)$; then take the infinite universe limit. You should find $I_\nu = B_\nu(T_\odot)$ and thus $I = B(T_\odot)$. A finite answer is obviously an improvement. But not much—in the naïve cosmology, the night sky is exactly as bright as the surface of the Sun! This catastrophe is the classic Olber's result.

(d) [**0.5 points**] Interpret your result from part (c) physically. What physically leads to Olber's paradox in the naïve universe? What effect(s) solve the paradox in a big-bang universe? You should have found that your answer was independent of $n_\star$ and the stellar radius adopted; why?

Finally, comment on the cosmological information encoded in the seemingly simple fact that the night sky is dark.

3. *The Surface Brightness of the Milky Way.* The technology you developed for Olber's paradox can also be applied to the Milky Way, which to zeroth approximation is a uniform disk of stars. Continue to assume a uniform density $n_\star$ of stars with radius $R_\odot$ and temperature $T_\odot$, and of frequency-independent stellar absorption with a geometrical cross section.

(a) [**0.5 points**] In the midplane of our Galaxy, we can only see a distance of about $s \sim 1$ kpc in any direction, due to absorption by dust. For a $s = 1$ kpc sightline, find the optical depth against absorption by stars, and show that the Galaxy is optically thin to absorption by stellar surfaces.

(b) [**0.5 points**] Use the solution you found in question 2c, but wavelength rather than frequency space. Show that your solution goes to $I_\lambda(s) \to \pi n_\star R_\odot^2 s\ B_\lambda(T_\odot)$ and thus $I(s) \to \pi n_\star R_\odot^2 s\ B(T_\odot)$ in the optically-thin limit. This is an estimate for the surface brightness of the Milky Way!

Evaluate $I_\nu(s)$ for $s = 1$ kpc, $T_\odot = 6000$ K, and $n_\star = 1\ \mathrm{pc}^{-3}$, at $\lambda = 530$ nm. Express your answer in cgs units erg cm^{-2} s^{-1} nm^{-1} sr^{-1}, and in SI units W m^{-2} nm^{-1} sr^{-1}.

(c) [**0.5 points**] Go to the ADS search engine found in the course links page. A relatively recent article on the (southern) Milky Way surface brightness is Hoffmann, B., Tappert, C., et al 1998 A&A Supplement, 128, 417. Figure 1b shows the Milky Way surface brightness in the V band which is centered on 530 nm. The plot uses mystical "S10" units, but following the link[1] in the Figure 1 caption, Table 3 gives the conversion from S10 into physical units.

[1] `http://cdsarc.u-strasbg.fr/vizier/ftp/cats/VII/199/ReadMe`

Compare your simple estimate to the observed Mikly Way surface brightness. Comment on the result.

(d) **[0.5 points]** Still using our simple "uniform disk" model, estimate the energy density of Milky Way starlight at our location. Express your answer in $\mathrm{eV/cm^{-3}}$.

4. *Multimessenger Radiation Transfer: Geoneutrinos.* The interior of the Earth is of course hotter than its surface. Moreover, this temperature difference (i.e., gradient) between the interior and exterior leads to a heat flux out of the Earth–this is flux of geothermal energy!

But where did the heat come from? Some may be left over from the formation of the Earth, but for sure some heat come from the decay of radioactivity in the Earth's interior. The most important decays are of the most abundant radioactive species, uranium, thorium, and the radioactive potassium isotope ^{40}K.

It turns out that all of these decays generate one or more neutrinos.[2] Amazingly, these "geoneutrinos" have now been *measured* by the KamLAND experiment in Japan (KamLAND Collaboration, 2011 Nature Geoscience, 4, 647), verifying that radioactive decays are indeed an important source of geothermal heat.

Current techniques are not sensitive to the *direction* of geoneutrino arrival. Thus, all that can be measured is the geoneutrino *flux*. However, it is of interest to envision futuristic experiments that *can* measure geoneutrino directionality and thus take resolved *images* of the Earth's interior. What will the "terrestrial hemisphere" (the analog of the celestial sphere) look like in neutrinos?

(a) **[1 point]** Take the Earth is a perfect sphere of radius $R_\oplus$, with a neutrino observer on the surface. Imagine that the Earth contains a uniform density of radioactivities throughout its interior, so that the (energy-integrated) neutrino emission coefficient is a constant j_0. Because neutrinos are so weakly interacting, to an *excellent* approximation you may take the Earth to be transparent to neutrinos and ignore absorption effects.

Find an expression for the angular dependence of the energy-integrated neutrino intensity $I(\theta)$, with θ the angle from the observer's nadir, i.e., the direction straight down.[3] *Hint:* you may find it useful to draw a diagram and find the sightline distance inside the Earth at a nadir angle θ.

Plot the angular dependence of $I(\theta)$ versus $\theta \in [0, \pi/2]$. In this "uniform density" model, do most neutrinos come through your feet or horizontally? Explain physically.

Comment on the information that could be gained from a future resolved image of the geoneutrino intensity pattern.

(b) **[1 bonus point]** It turns out that the "uniform density" model is likely incorrect. Rather, geologists expect that uranium and thorium are found in the Earth's outer layers (crust and mantle). Redo the calculation for a "hollow shell" model. That is, use a neutrino emission coefficient that is equal to j_0 in a shell that goes from some radius $R_{\rm shell}$ out to $R_\oplus$, and is zero at radii $< R_{\rm shell}$.

[2] For the experts, really these are *anti*neutrinos generated in β decays.

[3] Thus $\theta = 0$ neutrinos come up through your feet, while those at $\theta = \pi/2$ arrive horizontally.

Given an analytic expression for $I(\theta)$ in terms of parameters given, and plot the result for $R_{\rm shell} = 0.8 R_\oplus$. You should find there is a special angle θ; explain.

In this more realistic hollow shell model, from what direction to most neutrinos come? Why? What would the geoneutrino "glow" look like if you could see it?