Astronomy 501 Spring 2013
Problem Set #10
Due in class: Friday April 19
Total points: 7+0.5

1. *The Spectrum of a Classical Damped Oscillator.* Here we will consider a simplified classical model of the interaction of an atomic state with the electromagnetic field. We model the electron in an atom as a one-dimensional, nonrelativistic, classical damped harmonic oscillator. Here an electron with mass m_e oscillates with natural frequency ω_0, while experiencing a damping force with magnitude $m_e \Gamma \dot{x}$. The oscillator driven by an external electric field of strength $E(t) = E_0 \ e^{i\omega t}$. The resulting equation of motion is therefore

$$\ddot{x} = -\Gamma \dot{x} - \omega_0^2 x - \frac{eE_0}{m_e} \ e^{i\omega t} \tag{1}$$

   (a) **[0.5 points]** We can estimate the damping timescale by assuming that, in the absence of the external field, the radiation from the charge leads to a decay in the oscillator's amplitude. Assuming that the energy loss per oscillation is small compared to the oscillator's energy, show that the energy loss rate is given by the inverse of the loss timescale, and is

$$\Gamma = \frac{2}{3} \frac{e^2 \omega_0^2}{m_e c^3} \tag{2}$$

   Note that the numerical prefactor is somewhat arbitrary in our simplified treatment.

   (b) **[0.5 points]** Consider the motion in the absence of the driving field. Search for solutions of the form $x(t) = x_0 \ e^{i\alpha t}$, where as usual it is understood that the physical x coordinate is the real part of this quantity. Show that the resulting motion are damped oscillations; find the oscillation frequency and the damping timescale.

   (c) **[1 point]** Follow the discussion of a driven harmonic oscillator in pp. 100-102 of RL. Fill in the steps to their derivation to show that case, the cross section is

$$\sigma(\omega) = \sigma_{\rm T} \ \frac{\omega^4}{(\omega^2 - \omega_0^2) + (\omega_0 \Gamma)^2} \tag{3}$$

   with $\sigma_{\rm T}$ the Thompson cross section. Note the the τ parameter in RL is related to the width via $\Gamma = \omega_0^2 \tau$.

   Then go on to cross section near resonance, where $\omega \approx \omega_0$. Again following RL, show that this gives

$$\sigma(\omega) = \frac{\pi e^2}{m_e c} \frac{\Gamma}{(\omega - \omega_0)^2 + (\Gamma/2)^2} \tag{4}$$

$$\sigma(\nu) = \frac{\pi e^2}{m_e c} \frac{\Gamma}{4\pi^2 (\nu - \nu_0)^2 + (\Gamma/2)^2} \tag{5}$$

   and finally show that

$$\int_{-\infty}^{\infty} \sigma(\nu) \ d\nu = \frac{\pi e^2}{m_e c} \tag{6}$$

2. *Hydrogen and lithium in the Sun.*

   (a) **[0.5 points]** The Hα line of hydrogen is from the $3p \to 2s$ transition in atomic hydrogen. Using the Bohr formula, calculate the energy $E_{\mathrm{H}\alpha}$ and wavelength $\lambda_{\mathrm{H}\alpha}$ of this transition.

   Go online to the BASS2000 solar spectrum database.[1] Find the Hα line in the Sun, and estimate the full width at half maximum $\Delta\lambda_{\mathrm{FWHM}}$ of the line. Find the corresponding width in frequency space, $\Delta\lambda_{\mathrm{FWHM}} = (\Delta\nu_{\mathrm{FWHM}}/\nu_{\mathrm{H}\alpha})\lambda_{\mathrm{H}\alpha}$.

   (b) **[1 point]** Assume that the Hα line profile $\phi(\nu)$ at half-maximum and beyond is Lorentzian. Using this, find an expression for the optical depth τ_ν of the Hα line, in terms of physical constants as well as the oscillator strength $f_{3p\to 2s}$, natural width $\Gamma_{3p\to 2s}$, and the column density $N_{\mathrm{H(3p)}}$ of neutral hydrogen in the $3p$ state.

   Note that when the observed line strength is at half maximum, we have $e^{-\tau_\nu} = 1/2$. Using this, and the values $f_{3p\to 2s} = 0.64$ and $\Gamma_{3p\to 2s} = 5 \times 10^8\ \mathrm{s}^{-1}$, calculate the column density $N_{\mathrm{H(3p)}}$ in the solar photosphere. Express your answer in cm^{-2}.

   Finally, assuming the solar photosphere is in equilibrium at temperature $T_\odot = 5800$ K, find the ratio $n(1s)/n(3p)$ of atoms in the $1s$ and $3p$ states in the photosphere. Use this to calculate the column density $N_{\mathrm{H(1s)}}$ of hydrogen in the ground state.

   (c) **[1 point]** Now consider the neutral lithium (Li I) doublet at 6707.8Å and 6707.9Å. Find these lines in the solar spectrum, and explain why they are blended together.

   Estimate the maximum depth $\Delta F_\lambda/F_\lambda(0)$ of the Li I absorption.

   (d) **[1 point]** Using your result from part (c), and assuming thermal broadening, calculate the column density of Li I in the sun. Express your answer in cm^{-2}. Note that the 6707.8Å line has an oscillator strength of 0.498, while for the 6707.9Å line this is 0.249.

   (e) **[0.5 points]** Now use your results to find the solar ratio of Li I/H I. Comment on the implications of your result.

   (f) **[0.5 bonus points]** One can go on to find other abundances in the sun, e.g., that of Si. But both Li and Si are also found in meteorites. Comparing the two reveals that $(\mathrm{Li/Si})_\odot \ll (\mathrm{Li/Si})_{\mathrm{meteorites}}$. In other words, we find that lithium is depleted in the solar photosphere relative to the meteorites.

   Suggest a reason for this discrepancy. No peeking! *Hint:* if the Sun is at fault, what what might the lithium depletion tell us about solar physics?

3. **[1 point]** Rybicki and Lightman problem 10.5.

[1] URL `http://bass2000.obspm.fr/solar_spect.php`