Astronomy 501 Spring 2013
Problem Set #11: The Final Frontier

Due in class: Monday April 29

Total points: 7+1

1. *CO emission from giant molecular clouds.* The CO molecule is "polar" and thus has a permanent dipole moment. In addition, C and O are two of the most common astrophysical "metals," and the CO molecule is very tightly bound. For all of these reasons, CO provides one of the most important probes of molecular clouds and of star formation. Here we will model the rotational states of the CO molecule and use this to estimate the order of magnitude properties of molecular clouds.

   (a) [**0.5 points**] To model the rotational properties of the CO molecule, we will treat it as two masses m_a and m_b, separated by a distance r. The moment of inertia about the center of mass is thus $I = m_a r_a^2 + m_b r_b^2$. Show that this can be expressed as

   $$I = C\ m\ r^2 \tag{1}$$

   with $m = m_a m_b/(m_a + m_b)$ the reduced mass, and C a dimensionless constant. Find the value of C.

   (b) [**0.5 points**] For the system in part (a), find an expression for the rotational kinetic energy, written in terms of the molecule's angular momentum.

   Then use the quantum mechanical relation for angular momentum quantization, $L^2 = J(J+1)\hbar^2$, and express the rotational kinetic energy E_J in terms of the quantum number J.

   (c) [**0.5 points**] Consider an allowed dipole transition between two neighboring rotational levels J and $J-1$. Find expressions for the energy and frequency of photons arising form this transition.

   If many molecular rotational levels J are populated, what is the nature of the rotational spectrum? What is the nature of the rotational spectrum if the molecule is *not* polar, i.e., does not have a permanent electric dipole moment?

   (d) [**2 point**] Specialize to the important case of the $J = 1$ to $J = 0$ transition in carbon monoxide (CO). Specifically, consider the $^{12}C^{16}O$ molecule, with atomic masses $m(^{12}C) \approx 12m_p$, and $m(^{16}O) \approx 16m_p$. Given the laboratory value $\nu_{10} = 115.271208$ GHz, find the transition energy ΔE_{10} and the implies molecular separation r, expressing your result in Å. Comment on the reasonableness of your result.

   Calculate the excitation temperature $T_{ex} = \Delta E/k$. Will the CMB excite CO molecules into the $J = 1$ state?

   Finally, consider the molecule $^{13}C^{16}O$, made of the stable but much less abundant carbon isotope ^{13}C. Calculate the $J = 1$ to $J = 0$ transition frequency for $^{13}C^{16}O$, and compare it to that for the much more abundant $^{12}C^{16}O$. Is this difference observable? If so, why might it be useful to measure the CO transition for both isotopes?

(e) **[0.5 points]** Find a classical expression for a molecule's rotation frequency ω. Then use the quantum mechanical relation $L \sim \hbar$ to find an expression for the acceleration $\ddot{d}$ of the dipole moment d. Use this and the Larmor radiation power to find an expression for the Einstein A_{10} for the $J = 1$ to 0 transition.

Calculate A_{10}, using the measured CO dipole moment $d = 0.11$ Debye $= 0.023$ e Å. Compare to the laboratory value $A_{10} = 7.202 \times 10^{-8}\ \mathrm{s}^{-1}$, and comment.

(f) **[1 point]** Find an expression for the emissivity of the $J = 1$ to $J = 0$ transition in $^{12}C^{16}O$, written in terms of the total $n(\mathrm{CO})$ number density. Go on to find an expression for the intensity from this transition in the optically thin limit, in terms of the $N(\mathrm{CO})$ column density. You should find that both expressions depend on the fraction $n(\mathrm{CO}, J = 1)/n(\mathrm{CO})$ of CO molecules in in the $J = 1$ state.

(g) **[1 point]** The fraction of CO molecules in the $J = 1$ state is a result of he interaction of the CO molecules with the local radiation field, but also of CO collisions with other molecules. To get a sense for this competition, note that the collision rate per CO molecule is $\Gamma_{\mathrm{coll}} = n_c \langle \sigma_{\mathrm{coll}} v \rangle$, where n_c is the number density of collision partners, σ_{coll} is the collision cross section, and v is the relative velocity. Where the CO molecule is present, molecular hydrogen H_2 is usually the most abundant collision partner.

Explain why the dimensionless ratio $\Gamma_{\mathrm{coll}}/A_{10}$ should give a measure of the relative importance of collisional vs radiative excitation of CO. Go on to show that $\Gamma_{\mathrm{coll}}/A_{10} = 1$ defines a *critical density* $n_{c,\mathrm{crit}}$ of collision partners; find an expression for this critical density.

To see the importance of the critical density, consider the case where the radiation temperature is below that of the gas $T_{\mathrm{rad}} < T_{\mathrm{gas}}$. Which temperature will determine the fraction $n(\mathrm{CO}, J = 1)/n(\mathrm{CO})$ in systems where $n_c \ll n_{c,\mathrm{crit}}$? In systems where $n_c \gg n_{c,\mathrm{crit}}$?

For CO, find the critical density for H_2 assuming a geometrical collision cross section $\sigma_{\mathrm{coll}} \approx \pi r^2$ with $r \sim 1$ Å, and a thermal velocity with $T = 20$ K.

(h) **[1 point]** Show that if the CO states are in thermal equilibrium at temperature T, then the partition function is

$$Z = \sum (2J + 1) e^{-J(J+1)B/kT} \tag{2}$$

where the "rotation constant" $B = \hbar^2/2I$. Thus show that

$$\frac{n(\mathrm{CO}, J = 1)}{n(\mathrm{CO})} = \frac{3\ e^{-2B/kT}}{Z} \tag{3}$$

Evaluate Z and this fraction for $T = 20$ K; you should find that the sum in partition function converges after a modest number of terms. Are a significant portion of the CO molecules in the $J = 1$ state?

(i) **[1 point]** Consider a giant molecular cloud, with a typical mass of about $10^5\ M_\odot$, and a typical size of about 10 pc, with temperature $T = 20$ K. Assume that the cloud is dominantly made of molecular hydrogen H_2, with a CO abundances dominated by $^{12}C^{16}O$, and with the ratio $(CO/H_2)_{cloud} \approx (C/H)_\odot \simeq 10^{-4}$.

Assuming the cloud is optically thin, with a thermal velocity dispersion σ_v, calculate the $J = 1$ to 0 intensity through the center of the cloud, at the line center. If the cloud is at a distance $d = 10$ kpc, estimate the $J = 1$ to 0 flux at the line center. Express your answer in units of Jansky, where 1 Jy = 10^{-23} erg cm^{-2} s^{-1} Hz^{-1}. Current radio telescopes have sensitivity better than 1 milliJansky. Will the cloud be visible to such a telescope?

(j) [**1 bonus point**] We assumed the cloud was optically thin. Evaluate this assumption, and comment on the result.