Astronomy 501 Spring 2013
Problem Set #3
Due in class: Friday, Feb. 8
Total points: 7+1

1. *Kirchhoff Revisited.*

   (a) [**0.5 points**] Consider a central spherical blackbody at temperature $T_{\rm c}$, surrounded by a concentric, non-scattering shell that is thermally emitting at $T_{\rm s}$. For a slightline that passes from the central object through the shell, solve for I_ν as a function of $\tau_\nu(s)$. Write your solution in the form of $I_\nu = B_\nu(T_{\rm c}) + \Delta_\nu$, i.e., find $\Delta_\nu = I_\nu - B_\nu(T_{\rm c})$. For $T_{\rm c} > T_{\rm s}$, find I_ν and Δ_ν in the limits $\tau_\nu \ll 1$, $\tau_\nu \approx 1$, and $\tau_\nu \gg 1$, and comment. How will these change if $T_{\rm c} < T_{\rm s}$?

   (b) [**0.5 points**] Now consider the case in which the absorption coefficient α_ν is nonzero for a narrow region ("line") centered at ν_0 with width $\Delta\nu \ll \nu_0$. Also let the width have $\Delta\nu \ll kT_{\rm s}/h, kT_{\rm c}/h$, i.e., the line feature is narrow compared to the frequency scales over which the Planck spectra change. Using your solution for I_ν, explain how the spectrum I_ν will look for $T_{\rm c} > T_{\rm s}$ and for $T_{\rm c} < T_{\rm s}$.

   In the optical and infrared, the Sun's spectrum shows a continuum with absorption lines, but in the UV and X-rays it shows a continuum with emission lines. Interpret these physically on the basis of your results.

2. *The Rosseland Mean and Electron Scattering in the Sun*

   (a) [**0.5 points**] For fully ionized hydrogen, free electron scattering has a frequency-independent cross section $\sigma_{\rm T} = 8\pi e^4/3m_e^2c^4 = 0.665 \times 10^{-24}\ {\rm cm}^2$, Find the Rosseland mean $\alpha_{\rm R}$ absorption coefficient for the case where electron scattering is the only important extinction process. Also find the Rosseland mean opacity $\kappa_{\rm R}$ if medium has mass density ρ.

   (b) [**0.5 points**] Use the solar mass and radius to find the mean mass density of the Sun. For a uniform density, fully-ionized Sun, find the mean free path, in cm. Then find the optical depth for a photon at the center of the Sun. Finally, find the pathlength traveled by a photon on its random walk out of the Sun, and the time it takes for the photon to escape (expressed in a convenient unit, e.g., seconds or months or years). Comment on the implications of your results.

3. *Electromagnetic Radiation and Maxwell's Equations.* In class we looked at plane-wave solutions to Maxwell's equations in terms of their Fourier components. This problem is to show that many of the general properties of electromagnetic plane waves can be found without going to Fourier space. The results you will find will of course agree with those we found for each Fourier mode, but make no reference to specific modes and thus apply generally to *any* arbitrary wave train.

   This problem will also give you a chance to brush up on your vector identities.

(a) **[0.5 points]** Show that in vacuum, Maxwell's equations in Cartesian coordinates can be used to derive wave equations $\nabla^2\vec{E} - (1/c^2)\partial_t^2\vec{E} = 0$ and $\nabla^2\vec{B} - (1/c^2)\partial_t^2\vec{B} = 0$. *Hint:* you might want to take the curl of one or more of Maxwell's equations.

Now consider candidate solutions to these equations in the form $\vec{E}(\vec{r}, t) = \vec{E}(\hat{n}\cdot\vec{r} - ct)$ and $\vec{B}(\vec{r}, t) = \vec{B}(\hat{n}\cdot\vec{r} - ct)$ That is, each Cartesian component such as $E_i = E_i(\chi)$ of the fields is an *arbitrary* function of only one variable $\chi = \hat{n}\cdot\vec{r} - ct$ which combines the position and time. This implies, for example, that $\partial_t E_i = dE_i/d\chi\ \partial_t\chi = -cE_i'$, where $E_i' \equiv dE_i/d\chi$.

Show that $\vec{E}(\hat{n}\cdot\vec{r} - ct)$ and $\vec{B}(\hat{n}\cdot\vec{r} - ct)$ are solutions of the wave equation if $\hat{n}^2 = 1$, that is, $\hat{n}$ is a unit vector.

Show that solutions of this form represent plane waves propagating in the $\hat{n}$ direction with speed c. *Hint:* consider the behavior of $\vec{E}$ and $\vec{B}$ on surfaces of constant χ, first at a fixed time, and then at different times.

(b) **[1 points]** Use Maxwell's equations to show that $\vec{E}(\hat{n}\cdot\vec{r} - ct)$ and $\vec{B}(\hat{n}\cdot\vec{r} - ct)$ are both *transverse* waves. *Hint:* after using Maxwell's equations, you will probably want to integrate the results with respect to χ, which is equivalent to an integration with respect to time while holding position fixed. In doing so, you make take as a boundary condition that neither $\vec{E}$ nor $\vec{B}$ have spatially constant ("DC") components.

Also show that $\vec{B}(\chi) = \hat{n}\times\vec{E}(\chi)$ (in cgs units), and use this to show that $\vec{E}$, $\vec{B}$, and $\hat{n}$ are everywhere mutually *orthogonal*. Finally, show that $|\vec{E}(\chi)| = |\vec{B}(\chi)|$.

(c) **[0.5 points]** Show that the Pointing flux is

$$\vec{S}(\hat{n}\cdot\vec{r} - ct) = \frac{c}{4\pi}\ |E(\hat{n}\cdot\vec{r} - ct)|^2 \tag{1}$$

and that the energy density is

$$u(\hat{n}\cdot\vec{r} - ct) = \frac{1}{4\pi}\ |E(\hat{n}\cdot\vec{r} - ct)|^2 = \frac{|\vec{S}|}{c} \tag{2}$$

Note that these are *not* the same as the time-averaged values $\langle\vec{S}\rangle$ and $\langle u\rangle$ quoted in class. The time averaging is only trivial in the case of a monochromatic wave with a single frequency.

4. **[1 point]** Rybicki & Lightman. Problem 2.2

5. *Polarization*[1]

(a) **[1 point]** Consider a plane electromagnetic wave with major (1) and minor (2) axis electric fields given by

$$E_1 = \mathcal{E}_0 \cos\beta e^{-i\omega t} \tag{3}$$

and

$$E_2 = -\mathcal{E}_0 \sin\beta e^{-i\omega t - i\pi/2}. \tag{4}$$

Here

$$\mathbf{e}_1 = \mathbf{e}_x \cos\chi + \mathbf{e}_y \sin\chi \tag{5}$$

[1] Problems swiped from Charles Gammie.

$$\mathbf{e}_2 = -\mathbf{e}_x \sin\chi + \mathbf{e}_y \cos\chi \tag{6}$$

Show that for this wave

$$(I, Q, U, V) = \mathcal{E}_0^2(1, \cos 2\beta \cos 2\chi, \cos 2\beta \sin 2\chi, \sin 2\beta). \tag{7}$$

(b) [**0.5 points**] Consider a group of N linearly polarized electromagnetic wave, with polarization angles $2\pi n/N$, $n = 0, \ldots, N-1$. What is the polarization fraction?

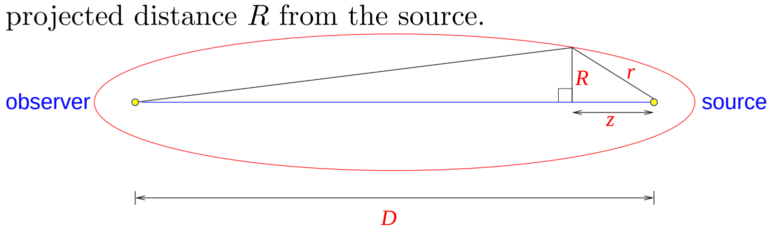
(c) [**0.5 points**] Consider a beam with Stokes $Q = 1$, $U = 0$. What would the Stokes parameters be if you rotated the beam by $90°$?

Then consider a plane wave with $V = 1$, $Q = U = 0$. In what sense does the beam rotate around the wavevector (and which are you using: the electrical engineering, or optics, convention)?

6. *Bonus Problem: Light Echoes.* In class we mentioned light echos as an interesting time-dependent application of optically thin scattering. Here we will explore this further.

Consider an observer and a transient source, separated by a distance D. Assume the environment around the source contains scattering sources, possibly inhomogeneous, but with small optical depth so that at most one scattering event will occur. The unscattered light will be seen as the transient outburst, and the scattered radiation will be the light echo.

As seen in the diagram below, at a given time, a point on the echo will be measured at a projected distance R from the source.



(a) [**0.5 bonus points**] If the source is observed a time t after the transient outburst is seen, show that the scattering sources lie on an *ellipse*, for which the source and the observer are each one focus. Also show that the major axis of the ellipse has length $D + ct$. *Hint*: the major axis is also the sum distances from the each focus to any point on the ellipse.

To identify the location of the scattering source at projected distance R, we must find its longitudinal distance z from the source (see diagram). Consider the "parabolic" approximation in which $D \gg r, z, R$. Show that in this approximation, we have

$$z = \frac{R^2}{2ct} - \frac{ct}{2} \tag{8}$$

which relates z to the observables R and t.

(b) [**0.5 bonus points**] For the case of SN 1987, the distance is $D \approx 50$ kpc. In the AAT light echo webpage in Lecture 8, there is a link to `a series of six images`. Look at the image from $t = 913$ days after the outburst. Estimate R for the two circles, and calculate z for each. What is the geometry of the scattering surfaces that produced the rings? Are they located in front of or behind the supernova?