**Astronomy 501 Spring 2013**
**Problem Set #6**
Due in class: Friday, March 8
Total points: 7+1

1. *Superluminal Motion.* Consider a blob of material ejected with speed $v < c$ relative to a supermassive black hole, and Lorentz factor $\gamma = (1 - v^2/c^2)^{-1/2}$. For simplicity, assume the black hole is at rest with respect to us, and ignore any gravitational or cosmological redshifting. As discussed in class, the motion of the blobs on the sky leads to apparent speeds $v_{\rm app} > c$: this is "superluminal motion."

   (a) **[1 point]** Let the blob move towards us at an angle $\theta$ with respect to the line of sight. Consider two epochs in which the blob emits. Show that the apparent speed is given by

   $$v_{\rm app} = \frac{v \sin\theta}{1 - (v/c)\cos\theta} \qquad (1)$$

   Hint: you may find it useful to find the distance $\Delta r_\perp$ the blob travels in the plane of the sky, and the difference $\Delta t_{\rm arr}$ in the arrival times of the signals from the two epochs.

   (b) **[1 point]** Use the result from part (a) to show that $v_{\rm app} \leq \gamma v$, with equality when $\cos\theta = v/c$.

   (c) **[1 bonus point]** Use the result from part (a) to show that $v_{\rm app} > c$ is only possible if $v > c/\sqrt{2}$.

   (d) **[1 point]** From the plot of the 3C279 jet, shown in the webpage from the Feb. 22 lecture, estimate $v_{\rm app}/c$. Find the *minimum* Lorentz factor $\gamma_0$ that consistent with your result, and the associated jet angle $\theta_0$.

   If the true Lorentz factor is $\gamma > \gamma_0$, will the true jet angle be larger than or smaller than $\theta_0$?

2. *Synchrotron Radiation.*[1] As discussed in class, radio galaxy jets arise from electron synchrotron radiation.

   (a) **[1 points]** Consider a relativistic electron with Lorentz factor $\gamma$ and relativistic energy $E = \gamma mc^2$, with $\gamma \gg 1$. The electron loses energy due to synchrotron radiation in a uniform magnetic field $B$. Show that we may write the energy loss as

   $$\dot{E} = -bE^2 \qquad (2)$$

   and find an expression for $b$.

   Also find an expression that estimates the timescale for energy loss.

   (b) **[0.5 points]** If an electron initially has energy $E_0$ at time $t = 0$, solve eq. (2) to find an expression for $E(t)$. At what time is all of the energy lost? Compare this to the timescale you found in part (a), and comment briefly.

[1] Portions swiped from a problem by Eugene Chiang.

(c) **[0.5 points]** Cosmic-ray electrons are observed at Earth with energies up to 1 TeV. Using typical values for the interstellar magnetic field, calculate the energy loss time for a 1 TeV electron. Express your answer in units of Gyr. Will cosmic-ray electrons with lower energies take more or less time to lose their energies?

Comment on the implications of your result.

(d) **[1 point]** Let's model the behavior of cosmic-ray electrons by finding the behavior of their energy spectrum. Let $N(E)\,dE$ give the number density of cosmic-ray electrons with energies $\in (E, E+dE)$. If we allow for cosmic-ray production (i.e., acceleration), and for energy loss due to synchrotron radiation, then the cosmic rays are governed by

$$\partial_t N + \partial_E \left(\dot{E}\ N\right) = q \qquad (3)$$

where $\dot{E} = -bE^2$ as found above, and where $q(E)\,dE$ is the number of cosmic ray electrons produced (i.e., accelerated or "injected") with energies $\in (E, E+dE)$. While this equation may look unfamiliar, it really is just a continuity equation in energy space, with $N$ playing the role of density and $\dot{E}$ playing the role of "velocity." Thus this equation is really just a statement of the conservation of electrons.

Thus, when cosmic rays are accelerated, their spectrum "at birth" is $N(E) \propto q(E)$, but this evolves with time due to energy losses.

Consider the "steady state" situation in which $\partial_t N = 0$. Show that the resulting cosmic-ray spectrum can be written in terms of $q(>E) = \int_E^\infty q(E')\ dE'$. Then specialize to the astrophysically interesting case where cosmic rays are accelerated with a power-law spectrum $q(E) = A\ E^{-\alpha}$. Show that in this case, the steady-state or "propagated" spectrum is also a power law, with index $p = \alpha+1$, i.e., one unit steeper than the acceleration spectrum.

Comment on why physically the propagated spectrum is steeper than the injection spectrum.

(e) **[1 point]** In radio galaxies, it is generally found that the spectral index $s$ (i.e., $F_\nu \propto \nu^{-s}$) of radio jet varies as a function of position along the jet. In particular, the spectral index goes to roughly $s \simeq 0.5$ at the edge of the jets where "hot spots" are found, and goes to $s \simeq 1$ at the galaxy where the jet originates.

How would you explain this trend based on your results from part (d)? Where along the jet are the "newest" electrons? That is, where are most of the electrons produced?