Astronomy 501 Spring 2013
Problem Set #7

Updated March 14 to fix typos and to add clarifications which are colored red.
Apologies for the errors!

Due in class: Friday, March 15

Total points: 7+1

1. *Inverse Compton Scattering.*

   (a) [**1 point**] Consider a collision between a photon and an electron. In the lab frame, the electron has relativistic energy $\gamma m_e c^2$, and the photon has energy ϵ, and the collision is *head-on.* Show that the lab-frame energy ϵ_1 of the Compton scattered photon is *maximum* when the scattering angle in the electron rest frame is $\theta = \pi$.[1] Also show that this maximum energy is

   $$\epsilon_{1,\max} = \frac{\gamma^2(1+v)^2\epsilon}{1+2\gamma(1+v)\epsilon/m_e c^2} \tag{1}$$

   To do this you will need to boost into the electron rest frame and then back into the lab frame.

   Finally, show trivially that in the limit of ultra-relativistic electrons

   $$\epsilon_{1,\max} \to \frac{4\gamma^2\epsilon}{1+4\gamma\epsilon/m_e c^2} \tag{2}$$

   and that if we are also in the Thompson limit $\gamma\epsilon \ll m_e c^2$, then the maximum upscattered energy is $\epsilon_{1,\max} \to 4\gamma^2\epsilon$.

   (b) [**1 point**] In class, we saw that the spectrum of inverse Compton emission for a power-law distribution $N(\gamma) = C\,\gamma^{-p}$ of electron energies takes the form

   $$j(\epsilon_1;\epsilon) = \sigma_{\rm T}\,\frac{du_{\rm ph}}{d\epsilon}\,C\int_0^\infty G(x)\;N(\gamma)\;d\gamma \tag{3}$$

   where $x = \epsilon_1/(4\gamma^2\epsilon) = \epsilon_1/\epsilon_{1,\max}$. The dimensionless spectral function $G(x)$ was given in class, and is peaked at $x_{\max} = 0.611$ or $\epsilon_1 = 4x_{\max}\gamma^2\epsilon$. Let's simplify the problem and assume that the spectral function is peaked as sharply as imaginable: $G(x) = \delta(x - x_{\max})$, with $x_{\max} = 0.611$. Find the emission function $j(\epsilon_1;\epsilon)$ in this approximation.[2] What is the dependence on ϵ_1 and on ϵ, and how do these compare with the results for the more careful solutions?

2. *Inverse Compton Scattering of Solar Photons.* In class we saw that the *Fermi* gamma-ray space telescope has recently measured inverse Compton scattering of solar radiation by cosmic-ray electrons. Here we will try to understand their basic result.

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[1] Originally this had $\theta = 0$ which is incorrect! Apologies!

[2] You may find it useful to recall the property of delta functions that $\int f(x)\,\delta[g(x)]\,dx = f(x_0)/|g'(x_0)|$ where x_0 is a zero of g, i.e., $g(x_0) = 0$, and $g' = dg/dx$.

(a) **[0.5 points]** Explain why inverse Compton scattering of solar photons is a particularly well-posed problem. That is, why should it be that the predictions we make are rather precise? *Hint:* Think about the main ingredients or inputs needed to calculate the inverse Compton emission, and then explain why these ingredients are particularly well-understood for this case.

(b) **[0.5 points]** Assuming the Sun is a blackbody, find the peak energy (related to the peak frequency) in the solar spectrum, and express your answer in eV. *Fermi* measures solar photons in the range ~ 100 MeV to ~ 100 GeV. Show that cosmic-ray electrons with $E_e = 10$ GeV will inverse-Compton scatter solar photons into the heart of the *Fermi* energy range.

(c) **[1 bonus point]** Consider a cosmic-ray electron inside the solar system. Show that the number density of solar photons scales as $n_\gamma \propto L_\odot/(\epsilon r^2)$, with ϵ the mean solar photon energy. Then show that a cosmic ray on a radial trajectory will encounter an "optical depth" against Compton scattering is

$$\tau \sim \frac{\sigma_{\rm T}\, L_\odot}{4\pi\epsilon c R_\odot} \tag{4}$$

and compute the value of τ. Will most cosmic-ray electrons in the heliosphere undergo Compton scattering?

(d) **[1 point]** Near the Earth, the flux of cosmic-ray electrons with energies of 10 GeV and above is about $\Phi_e \simeq 10^{-4}$ electrons cm^{-2} s^{-1}. Assuming these are all on radial trajectories, show that the inverse Compton luminosity of the Sun is about $L_{\rm IC} \sim 4\pi a^2 \Phi_e \tau$. Using this, find the inverse Compton flux from the Sun.

Compare your result to the *Fermi* measurement, A. A. Abdo et al, 2011 ApJ 734, 116. Comment on the agreement.

3. *The Sunyaev-Zeldovich Effect.*[3] For repeated inverse Compton scattering by nonrelativistic electrons with a temperature T_e *which we will treat as fixed*, the change in photon occupation number f is given by the Kompaneets equation which describes how f changes in frequency ν and time t. This equation in its original form is

$$\frac{\partial}{\partial t} f = \frac{n_e \sigma_{\rm T}}{m_e c} \frac{h}{\nu^2} \frac{\partial}{\partial \nu} \left[\nu^4 \left(\frac{kT_e}{h} \frac{\partial f}{\partial \nu} + f + f^2 \right) \right] \tag{5}$$

This can be re-expressed in terms of a dimensionless variable $x = h\nu/kT_e$ as

$$\frac{\partial}{\partial t} f = n_e \sigma_{\rm T} c\, \frac{kT_e}{m_e c^2} \frac{1}{x^2} \frac{\partial}{\partial_x} \left[x^4 \left(\frac{\partial f}{\partial x} + f + f^2 \right) \right] \tag{6}$$

Recall that the occupation number is defined such that the number of photons per unit volume with frequency in $(\nu, \nu + d\nu)$ is $dn = 8\pi\nu^2\, f/c^3\, d\nu$.

(a) **[1 point]** Before we use the full Kompaneets equation, let's first consider the incident photons, prior to scattering. Let these have a blackbody spectrum with

$$f_0 = \frac{1}{e^{h\nu/kT_{\rm rad}} - 1} \tag{7}$$

[3] Swiped from a problem by Wayne Hu.

where $T_{\rm rad} \ll T_e$, and for the purposes of this problem we also treat the $T_{\rm rad}$ as a fixed constant.

Show that for this blackbody spectrum, we have

$$f_0 + f_0^2 = -\frac{T_{\rm rad}}{T_e}\frac{\partial f_0}{\partial x} \tag{8}$$

(b) **[1 point]** Turning to the Kompaneets equation, change variables from t to the Compton-y parameter, where

$$dy = n_e \sigma_{\rm T} c \, \frac{k(T_e - T_{\rm rad})}{m_e c^2} \, dt \tag{9}$$

where the electron density n_e is taken as a fixed constant.

We are interested in the case where Compton scatterings make a small perturbation to the incident blackbody photons. That is, wish to find the deviations $\Delta f = f - f_0$ which we assume are small. To do this, show that we can use the result from part (a) to write the Kompaneets equation as

$$\frac{\partial}{\partial y} f \approx \frac{1}{x^2}\frac{\partial}{\partial_x}\left(x^4 \frac{\partial f}{\partial x}\right) = \frac{1}{x_\nu^2}\frac{\partial}{\partial_{x_\nu}}\left(x_\nu^4 \frac{\partial f}{\partial x_\nu}\right) \tag{10}$$

where $x_\nu = h\nu/kT_{\rm rad}$.

(c) **[1 point]** Assuming that the deviations $\Delta f = f - f_0$ are small, we can substitute $f_0(x_\nu)$ into the righthand side of eq. (10). The integration is then trivial; show that it gives

$$\frac{\Delta f}{f_0} = -y \; x_\nu \; \frac{e^{x_\nu}}{e^{x_\nu} - 1}\left(4 - x_\nu \frac{e^{x_\nu} + 1}{e^{x_\nu} - 1}\right) \tag{11}$$

For small perturbations, $\Delta f / f_0 \approx \Delta I_\nu / I_\nu$. What is $\Delta I_\nu / I_\nu$ as $x_\nu \to 0$? $x_\nu \to \infty$? That is, find the leading nonzero term in each limit.

Numerically find the value of x_ν at which $\Delta I_\nu / I_\nu = 0$. This is known as the SZ null. For $T_{\rm rad} = 2.725$ K, find the null frequency in GHz.