Astronomy 501 Spring 2013
Problem Set #9
Due in class: Monday April 8
Total points: 7+1

1. *A Thermodynamic Warmup.* In class we found that in the non-relativistic, non-degenerate limit, the number density of a particle of mass m is

$$n = g\ n_{\rm Q}\ e^{-(mc^2-\mu)/kT} \quad (1)$$

where g counts the particle's internal degrees of freedom, and

$$n_{\rm Q} = \left(\frac{2\pi mkT}{h^2}\right)^{3/2} \quad (2)$$

is the quantum concentration.

(a) **[0.5 point]** The thermal de Broglie wavelength Λ is the de Broglie wavelength of a particle with thermal velocity $v_T^2 = kT/m$. Shw that the quantum concentration can be written as $n_{\rm Q} = \alpha\ \Lambda^3$, and find the value of α.

Given this relation, what differences would you expect between matter with density $n \gg n_{\rm Q}$ compared to matter with $n \ll n_{\rm Q}$?

(b) **[1 point]** Estimate the number density of nitrogen molecules N_2 (atomic weight ≈ 28) in air at room temperature. *Hint*: air has a mass density $\rho_{\rm air} \sim 10^{-3}$ g/cm^3. Then find the quantum concentration for nitrogen molecules at room temperature.

A gas with $n \ll n_{\rm Q}$ is non-degenerate, and thus its equation of state is well approximated by the ideal gas law $P = n\ kT$. That is, quantum effects are not important in determining the equation of state. In the air in the room well-approximated as an ideal gas?

Finally, find the chemical potential of air in the room, expressing your answer in terms of the ratio $\mu_e'/kT = (\mu_e - m_e c^2)/kT$.

(c) **[0.5 points]** The density of solid iron (atomic weight ≈ 56, atomic number $Z = 26$) is about $\rho_{\rm Fe} \sim 5$ g/cm^3. Find the number density of *electrons* in solid iron. Then find the quantum concentration of *electrons* at room temperature. Compare the two, and comment on the result.

2. *Ionization in the Solar Photosphere.*

(a) **[0.5 points]** The solar photosphere is the "surface of last scattering" for photons emerging from the Sun. Explain why we may estimate the photospheric properties by assuming that the photosphere is where the mean free path against Thompson scattering is $\sim R_\odot$, the observed solar radius. Using this relation, estimate the free electron number density n_e at the solar photosphere; express your answer in cm^{-3}.

(b) **[0.5 points]** Use the Saha equation find an expression for the neutral hydrogen density $n_{\rm H}$ in the photosphere to the free electron density n_e and the photospheric temperature $T_\odot$. You may assume that the Sun is entirely made of hydrogen.

(c) **[0.5 points]** Using $T_\odot = 5800$ K, use your expression to estimate the ionization fraction of the photosphere. Is the photosphere mostly ionized or mostly neutral?

Comment on the consequences of this result for observing hydrogen lines in the solar spectrum.

3. *Saha and Cosmic Recombination.* As seen in class, we can use the Saha equation to estimate the temperature and redshift of (re)combination in the early Universe.

   (a) **[0.5 points]**. As a starting point, we will take as given the important cosmological result[1] that the cosmic number density of protons (either bound or free) is roughly $n_{p,\rm tot} \approx \eta n_\gamma$. Here $\eta \equiv n_{\rm b}/n_\gamma$ is the ratio of cosmic baryons to photons, and $n_\gamma(T)$ is the cosmic photon number density, namly that of a blackbody at temperature T.

   Using this, and assuming the universe is made only of hydrogen, show that the Saha equation can be used to write the electron fraction x_e as a function only of temperature T and given constants. Write this expression in terms of the variable $U = kT/B$.

   (b) **[1 point]**. One definition of recombination is $x_e = 1/2$. Using this, solve for U. To do this you will either need to use a computer, or to use iteration. The naïve estimate of the recombination temperature is $kT_{\rm rec,naive} = B$, or $U = 1$. Is your result larger or smaller than the naïve estimate? Briefly explain why.

   Using your result, find the corresponding recombination redshift $z_{\rm rec}$, defined by $T_{\rm rec} = (1 + z_{\rm rec})T_0$, with the present CMB temperature $T_0 = 2.725$ K.

   (c) **[1 bonus point]**. Our definition of recombination as $x_e = 1/2$ is reasonable but arbitrary. Find the value of U for $x_e = 0.1$, and also for $x_e = 0.01$. For each of these cases, find the corresponding recombination redshift $z_{\rm rec}$. Comment on your result, and give your estimate of the redshift range over which recombination occurs.

4. **[2 points]**. Rybicki & Lightman problem 10.2; each part is worth 0.2 points.

[1] If you haven't already, please take the Physical Cosmology course to see where these results come from!