Astro 501: Radiative Processes

Lecture 10

Feb 6, 2013

Announcements:

- **Problem Set 3** due next time
- BDF office hours shortened today
- TA office hours 1:30-3:00 tomorrow

Last time: began classical EM radiation

Q: energy density?

Q: Poynting vector?

Today: plane waves & polarization

Maxwell and Fourier Modes

We have seen: wave equation demands $\omega = ck$
But Maxwell equations impose further constraints

Consider arbitrary Fourier modes
$\vec{E} = E_0 \ e^{i(\vec{k}\cdot\vec{r}-\omega t)} \ \hat{a}_1$, and $\vec{B} = B_0 \ e^{i(\vec{k}\cdot\vec{r}-\omega t)} \ \hat{a}_2$

Maxwell equations in vacuum impose conditions:
for example, Coulomb's law $\nabla \cdot \vec{E} = 0$ implies

$$\vec{k} \cdot \vec{E} = 0 \tag{1}$$

or equivalently $\hat{n} \cdot \hat{a}_1 = 0$

similarly, no monopoles requires

$$\vec{k} \cdot \vec{B} = 0 \qquad \hat{n} \cdot \hat{a}_2 = 0 \tag{2}$$

Q: what does this mean physically for the waves?

we found $\vec{k} \cdot \vec{E} = \vec{k} \cdot \vec{B} = 0$

$\rightarrow$ propagation orthogonal to field vectors

$\Rightarrow$ *EM waves are transverse*

Faraday's law requires $\omega \vec{B} = c\vec{k} \times \vec{E}$, or

$$\vec{B} = \frac{c\vec{k}}{\omega} \times \vec{E} = \hat{n} \times \vec{E} \tag{3}$$

and Ampère's law gives $\vec{E} = -\hat{n} \times \vec{B}$

Q: what do these conditions imply for the waves?

Faraday's law gives $\vec{B} = \hat{n} \times \vec{E}$, so

$$\vec{E} \cdot \vec{B} = \vec{E} \cdot (\hat{n} \times \vec{E}) = 0 \qquad (4)$$

$\Rightarrow$ $\vec{E}$ *and* $\vec{B}$ *are orthogonal to each other!*

Faraday also implies

$$|B|^2 = \hat{n}^2 |E|^2 - |\hat{n} \cdot \vec{E}|^2 = |E|^2 \qquad (5)$$

using vector identity $(\hat{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = \hat{a} \cdot \vec{c}\, \vec{b} \cdot \vec{d} - \hat{a} \cdot \vec{d}\, \vec{b} \cdot \vec{c}$

we have: $E_0 = B_0$: *field amplitudes are equal*

which in turn means: $\hat{a}_2 = \hat{n} \times \hat{a}_1$, and $\hat{a}_1 \cdot \hat{a}_2 = 0$
$\rightarrow$ $(\hat{n}, \hat{a}_1, \hat{a}_2)$ form an *orthogonal basis*

Monochromatic Plane Wave: Time Averaging

at a given point in space, field amplitudes vary
sinusoidally with time $\to$ energy density and flux also sinusoidal
but we are interested in timescales $\gg \omega^{-1}$:
$\to$ take *time averages*

Useful to use *complex* field amplitudes
then take *real part* to get physical component

handy theorem: for $A(t) = \mathcal{A}e^{i\omega t}$ and $B(t) = \mathcal{B}e^{i\omega t}$
i.e., same time dependence, then time-averaged products

$$\langle \mathrm{Re}A(t)\ \mathrm{Re}B(t) \rangle = \frac{1}{2}\mathrm{Re}(\mathcal{A}\mathcal{B}^*) = \frac{1}{2}\mathrm{Re}(\mathcal{A} * \mathcal{B}) \qquad (6)$$

Monochromatic Plane Wave: Energy, Flux

time-averaged Poynting flux amplitude

$$\langle S \rangle = \frac{c}{8\pi} \mathrm{Re}(E_0 B_0^*) = \frac{c}{8\pi} |E_0|^2 = \frac{c}{8\pi} |B_0|^2 \tag{7}$$

time-averaged energy density

$$\langle u \rangle = \frac{c}{8\pi} |E_0|^2 = \frac{c}{8\pi} |B_0|^2 \tag{8}$$

Q: given wave direction $\vec{n}$, degrees of freedom in $\vec{E}, \vec{B}$?

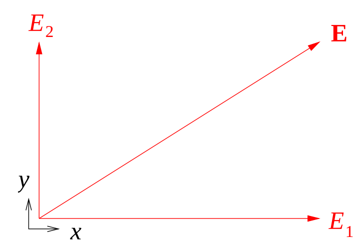
Polarization

EM waves propagating *in a particular direction* $\hat{n}$
must be transverse $\vec{k} \cdot \vec{E} = \hat{n} \cdot \vec{E} = 0$
$\rightarrow$ nonzero $\vec{E}$ *components lie in plane* $\perp$ to $\hat{n}$

two independent components

for convenience: choose coordinates where $\hat{n} = \hat{z}$
so fields are in transverse plane $x - y$
physical electric vector is *real part* of

$$\vec{E} = (E_1 \; \hat{x} + E_2 \; \hat{y}) \; e^{-i\omega t} \tag{9}$$



complex amplitudes can be written

$$E_1 = \mathcal{E}_1 \; e^{i\phi_1} \quad E_2 = \mathcal{E}_2 \; e^{i\phi_2} \tag{10}$$

Q: but wait–what about the magnetic field?

transverse electric field has two independent components
but once $\vec{E}$ determined, then $\vec{B} = \hat{n} \times \vec{E}$
at every point along sightline $\hat{n}$, magnetic $\perp$ electric
$\Rightarrow$ *no additional degrees of freedom for* $\vec{B}$

monochromatic plane wave *has two independent components*

consider plane at fixed $z = \hat{n} \cdot \vec{r}$, say $z = 0$
the two *physical* components of the field evolve as

$$E_x = \mathcal{E}_1 \ \cos(\omega t - \phi_1) \quad E_y = \mathcal{E}_2 \ \cos(\omega t - \phi_2) \tag{11}$$

with E_1, E_2 can take any values, and ϕ_1, ϕ_2 independent
but only difference $\phi_1 - \phi_2$ can be important
$\rightarrow$ a total of *3 independent parameters* describe the fields

∞ *Q:* $\vec{E}$ *time evolution if* E_1 *and* E_2 *can differ, but* $\phi_1 - \phi_2 = 0$*?*
Q: same but $\phi_1 - \phi_2 = \pi$*?*

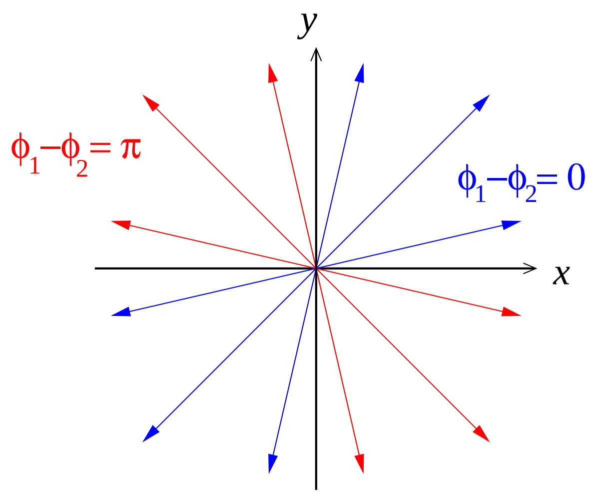
Linear Polarization

For $\phi_1 - \phi_2 = 0$, we have

$$E_x = \mathcal{E}_1 \ \cos(\omega t - \phi_1) \quad E_y = \mathcal{E}_2 \ \cos(\omega t - \phi_1) = \frac{\mathcal{E}_2}{\mathcal{E}_1} E_x \qquad (12)$$

fields share same sign and same sinusoidal time dependence
$\vec{E}$ sweeps out *line with positive slope* in $x - y$ plane
$\rightarrow$ **linear polarization**

For $\phi_1 - \phi_2 = \pi$, fields share time dependence
but have opposite sign
$\rightarrow$ linear polarization with negative slope

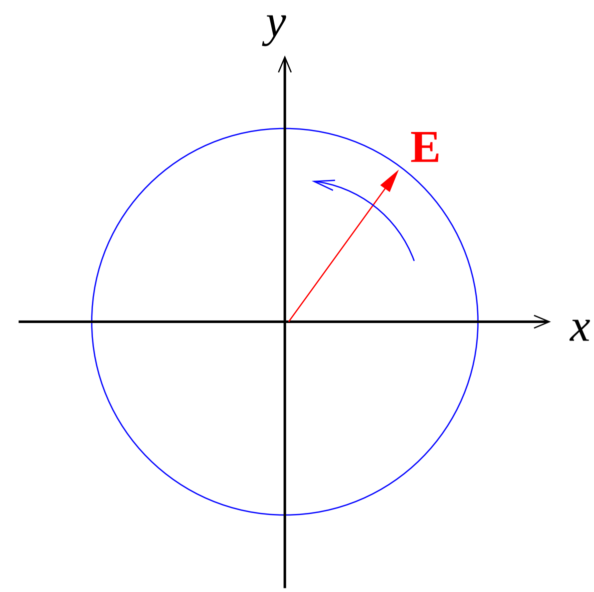


Q: what is $\vec{E}$ time dependence if
$\mathcal{E}_1 = \mathcal{E}_2$ but $\phi_1 - \phi_2 = \pi/2$? $-\pi/2$

Circular Polarization

if $\mathcal{E}_1 = \mathcal{E}_2$ but $\phi_1 - \phi_2 = \pi/2$

$$E_x = \mathcal{E}_1 \cos(\omega t - \phi_1) \quad E_y = \mathcal{E}_1 \sin(\omega t - \phi_1)$$



$\vec{E}$ sweeps counterclockwise circle
as seen approaching observer

$\Rightarrow$ **circular polarization**

Engineering: "*lefthanded*" circular polarization

$\rightarrow$ but using righthand rule: *positive helicity*

if $\mathcal{E}_1 = \mathcal{E}_2$ but $\phi_1 - \phi_2 = -\pi/2$

$\rightarrow$ "*righthand*" circular polarization, or *negative helicity*

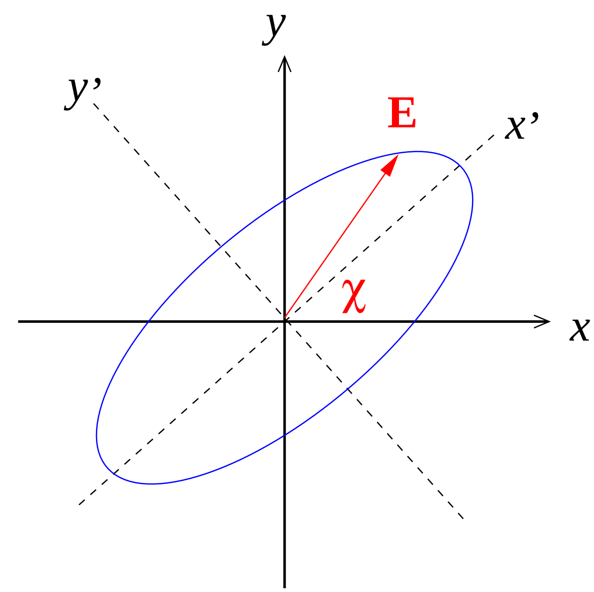
in the most general case: $\mathcal{E}_1 \neq \mathcal{E}_2$ and $\phi_1 - \phi_2$ arbitrary

Q: what is $\vec{E}$ time dependence?

Elliptical Polarization

in the general case

$$E_x = \mathcal{E}_1 \ \cos(\omega t - \phi_1) \quad E_y = \mathcal{E}_2 \ \cos(\omega t - \phi_2)$$



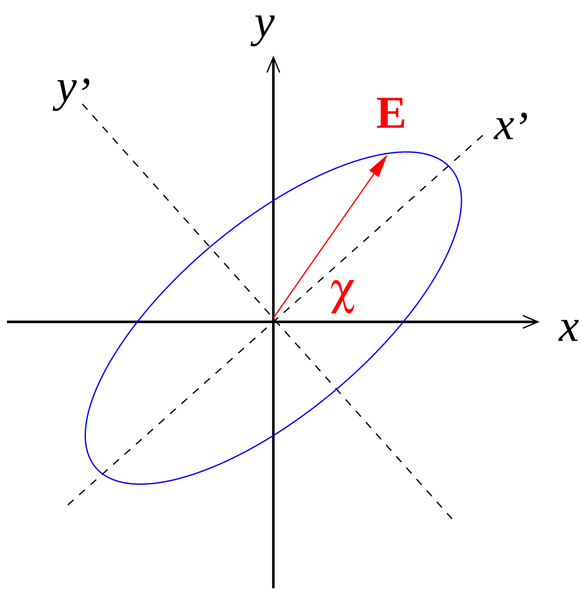
intuitively, blends linear and circular features:
$\rightarrow$ **elliptical polarization**

ellipse *orientation* fixed by $\mathcal{E}_1 - \mathcal{E}_2$ difference
ellipse *eccentricity* and *helicity* fixed by $\phi_1 - \phi_2$ difference

in coordinates (x', y') rotated to align with *principal axes*

$$E'_x = \mathcal{E}_0 \ \cos\beta \ \cos(\omega t) \quad E'_y = \mathcal{E}_0 \ \sin\beta \ \sin(\omega t)$$

for some $\beta \in [-\pi/2, +\pi/2]$
Q: evolution if $\beta > 0$?



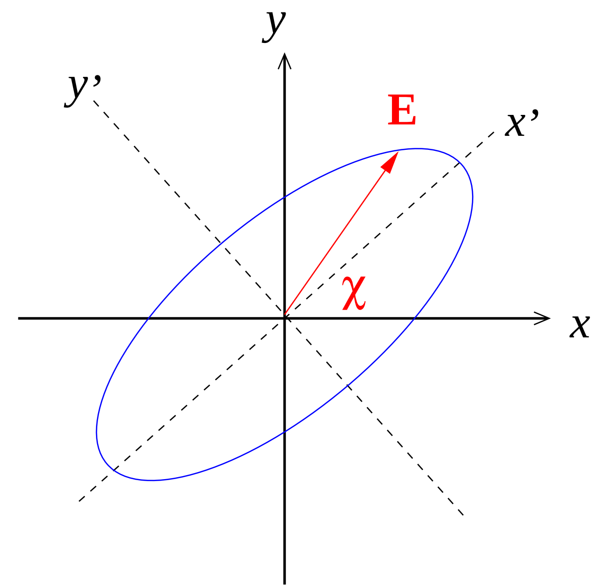
$$E'_x = \mathcal{E}_0 \ \cos\beta \ \cos(\omega t) \qquad E'_y = -\mathcal{E}_0 \ \sin\beta \ \sin(\omega t)$$

principle axes: $\mathcal{E}_0 \ \cos\beta$ and $\mathcal{E}_0 \ \sin\beta$

if $\beta \in [0, \pi/2]$: ellipse sweeps clockwise
$\rightarrow$ "*righthanded*" elliptical polarization, *negative helicity*
if $\beta \in [0, \pi/2]$: *"lefthanded", positive helicity*

Q: what give linear polarization? circular?

we want to relate $x - y$ field parameters $\mathcal{E}_1, \mathcal{E}_2, \phi_1, \phi_2$
with principle axes $x' - y'$ parameters $\mathcal{E}_0, \beta, \chi$
rotate $x - y$ components by angle χ



$$
\begin{aligned}
E_x &= \mathcal{E}_0 \left(\cos\beta \cos\chi \cos\omega t - \sin\beta \sin\chi \sin\omega t \right) \\
E_y &= \mathcal{E}_0 \left(\cos\beta \sin\chi \cos\omega t - \sin\beta \cos\chi \sin\omega t \right)
\end{aligned}
$$

matching to, e.g., $E_x = \mathcal{E}_1 \cos(\omega t - \phi_1)$:

$$\mathcal{E}_1 \cos\phi_1 = \mathcal{E}_0 \cos\beta \cos\chi \tag{13}$$

$$\mathcal{E}_1 \sin\phi_1 = \mathcal{E}_0 \sin\beta \sin\chi \tag{14}$$

$$\mathcal{E}_2 \cos\phi_2 = \mathcal{E}_0 \cos\beta \sin\chi \tag{15}$$

$$\mathcal{E}_2 \sin\phi_2 = -\mathcal{E}_0 \sin\beta \cos\chi \tag{16}$$

Q: how can we determine polarization by intensity measurements?

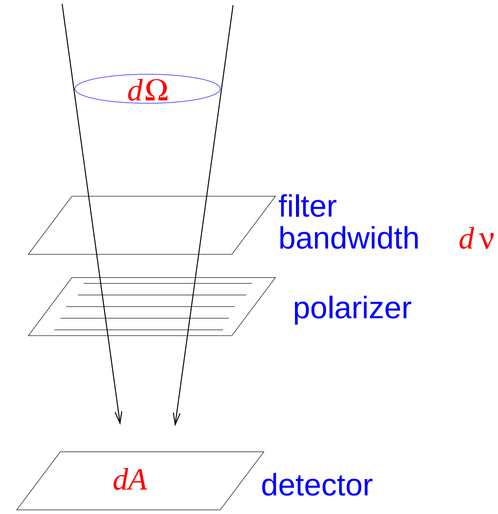
Introduce *polarizer*

can *rotate* polarizer:

$\rightarrow$ measure I_x, I_y, and 45° rotated $I_{x'}, I_{y'}$

can use circular polarizers to measure

$\rightarrow$ positive and negative circular polarization I_+, I_-



combine: **Stokes parameters**

$$I = I_x + I_y \quad (17)$$
$$Q = I_x - I_y \quad (18)$$
$$U = I_{x'} - I_{y'} \quad (19)$$
$$V = I_+ - I_- \quad (20)$$

Q: what physically is each? can more than one of Q, U, V be nonzero? what does that correspond to?

Q: range of values for Q? U? V? are they all independent?

Stokes Parameters

for *monochromatic waves*, Stokes parameters related to $\mathcal{E}_1, \mathcal{E}_2, \phi_1, \phi_2$ and $\mathcal{E}_0, \beta, \chi$ bases:

$$
\begin{aligned}
I &= \mathcal{E}_1^2 + \mathcal{E}_2^2 = \mathcal{E}_0^2 && (21)\\
Q &= \mathcal{E}_1^2 - \mathcal{E}_2^2 = \mathcal{E}_0^2 \cos 2\beta \cos 2\chi && (22)\\
U &= 2\mathcal{E}_1\mathcal{E}_2 \cos(\phi_1 - \phi_2) = \mathcal{E}_0^2 \cos 2\beta \sin 2\chi && (23)\\
V &= 2\mathcal{E}_1\mathcal{E}_2 \sin(\phi_1 - \phi_2) = \mathcal{E}_0^2 \sin 2\beta && (24)
\end{aligned}
$$

and thus

$$
\begin{aligned}
\mathcal{E}_0 &= \sqrt{I} && (25)\\
\sin 2\beta &= V/I && (26)\\
\tan 2\chi &= U/Q && (27)
\end{aligned}
$$

since wave has 3 independent parameters,
Stokes parameters must be *related*

$$I^2 = Q^2 + U^2 + V^2 \tag{28}$$

Quasi-Monochromatic Waves

natural light generally not a pure monochromatic wave
with a single, definite, complete state of polarization

rather: a *superposition* of components with many polarizations

consider wave with *slowly varying* amplitudes and phases

$$E_1(t) = \mathcal{E}_1(t)\ e^{i\phi_1(t)}\ ; \quad E_2(t) = \mathcal{E}_2(t)\ e^{i\phi_2(t)} \tag{29}$$

"slow": wave looks completely polarized on timescalse ω^{-1}
but amplitudes and phases drift over intervals $\Delta t \gg \omega^{-1}$
$\rightarrow$ polarization changes

but also wave is *no longer monochromatic*
frequency spread: "*bandwidth*" $\Delta\omega \sim 1/\Delta t \ll \omega$
$\rightarrow$ *quasi-monochromatic wave*

Q: effect on Stokes?

Stokes Parameters for Quasi-Monochromatic Light

real measurements represent averages over timescales during which polarization can change

Stokes parameters become averages

$$I = \langle E_1 E_1^* \rangle + \langle E_2 E_2^* \rangle = \langle \mathcal{E}_1^2 + \mathcal{E}_2^2 \rangle \tag{30}$$

$$Q = \langle E_1 E_1^* \rangle - \langle E_2 E_2^* \rangle = \langle \mathcal{E}_1^2 - \mathcal{E}_2^2 \rangle \tag{31}$$

$$U = \langle E_1 E_2^* \rangle + \langle E_2 E_1^* \rangle = 2 \langle \mathcal{E}_1 \mathcal{E}_2 \cos(\phi_1 - \phi_2) \rangle \tag{32}$$

$$V = -i \left(\langle E_1 E_2^* \rangle - \langle E_2 E_1^* \rangle \right) = 2 \langle \mathcal{E}_1 \mathcal{E}_2 \sin(\phi_1 - \phi_2) \rangle \tag{33}$$

but for quasi-monochromatic waves

$$I^2 \geq Q^2 + U^2 + V^2 \tag{34}$$

- quasi-monochromatic polarization is still in general *elliptical*
- but drifts can reduce degree of polarization

$$I^2 \geq Q^2 + U^2 + V^2 \tag{35}$$

- maximum polarization when equality holds: *completely elliptically polarized*
- minimum when $Q = U = V = 0$: *unpolarized*
- arbitrary wave is *partially polarized*

useful to define *polarized* intensity

$$I_{\mathrm{pol}} = Q^2 + U^2 + V^2 \tag{36}$$

and since $I_{\mathrm{pol}} \leq I$, define fractional **degree of polarization**

$$\Pi \equiv \frac{I_{\mathrm{pol}}}{I} = \frac{\sqrt{Q^2 + U^2 + V^2}}{I} \tag{37}$$

note: can always decompose Stokes parameters

$$\begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \begin{pmatrix} I - I_{\rm pol} \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} I_{\rm pol} \\ Q \\ U \\ V \end{pmatrix} \tag{38}$$

sum of unpolarized and polarized components

Superposition and Stokes

consider composite wave that is superposition of many independent waves
electric field components are given by **superposition**

$$E_1 = \sum_k E_1^{(k)} \quad ; \quad E_2 = \sum_k E_1^{(k)} \tag{39}$$

each term k of which has different phase

PS3: phases specified, can calculate sum explicitly

but generally, *phases are random*
so field products average out phases from different waves

$$\langle E_i E_j^* \rangle = \sum_k \sum_\ell \langle E_i^{(k)} E_j^{(\ell)*} \rangle = \sum_k \langle E_i^{(k)} E_i^{(k)*} \rangle \tag{40}$$

but due to this averaging, *Stokes parameters are additive*

$$I = \sum_k I^{(k)} \tag{41}$$

$$Q = \sum_k Q^{(k)} \tag{42}$$

$$U = \sum_k U^{(k)} \tag{43}$$

$$V = \sum_k V^{(k)} \tag{44}$$