Astro 501: Radiative Processes
Lecture 11
Feb 8, 2013

Announcements:
- **Problem Set 3** due at 5pm today in TA mailbox
- **Problem Set 4** due next Friday

Last time: plane waves & polarization
Q: most general polarization state of monochromatic light?
Q: for natural light?
Q: why the difference?

Today: potentials, and radiation by accelerated charges

The Vector Potential

No-molopoles condition $\nabla \cdot \vec{B}$
strongly restricts $\vec{B}$ configurations

condition *automatically* satisfied if we write

$$\vec{B} = \nabla \times \vec{A} \tag{1}$$

guarantees zero divergence because, for *any* $\vec{A}$

$$\nabla \cdot (\nabla \times \vec{A}) = 0 \tag{2}$$

where $\vec{A}$ is the **vector potential**
Q: units of $\vec{A}$*?*

write Faraday's law in terms of $\vec{A}$:

$$\nabla \times \vec{E} = -\frac{1}{c}\partial_t(\nabla \times \vec{A}) \tag{3}$$

and so

$$\nabla \times \left(\vec{E} + \frac{1}{c}\partial_t\vec{A}\right) = 0 \tag{4}$$

strongly restricts $\vec{E}$ configurations

Q: how to automatically satisfy?

The Scalar Potential

Faraday with $\vec{A}$

$$\nabla \times \left(\vec{E} + \frac{1}{c} \partial_t \vec{A} \right) = 0 \tag{5}$$

vector field $\vec{E} + \frac{1}{c} \partial_t \vec{A}$ is curl-free

to automatically satisfy this, note that

$$\nabla \times (\nabla \phi) = 0 \tag{6}$$

curl of grad vanishes for any scalar field (=function) ϕ

define **scalar potential** via

$$\vec{E} = -\nabla \phi - \frac{1}{c} \partial_t \vec{A} \tag{7}$$

Q: units of ϕ?

Q: are $\vec{A}$ and ϕ unique? why?

Gauge Freedom

vector potential defined to give $\nabla \times \vec{A} = \vec{B}$
clearly if $\vec{A} \to \vec{A}' = \vec{A} +$ constant, $\vec{B} \to \vec{B}$
$\Rightarrow$ physical field unchanged

in fact: $\vec{B}$ unchanged for *any transformation*
$\vec{A} \to \vec{A}'$ which preserves $\nabla \times \vec{A}' = \vec{B}$:

$$\nabla \times (\vec{A}' - \vec{A}) = 0 \tag{8}$$

and thus there is no physical change if

$$\vec{A}' = \vec{A} + \nabla\psi \tag{9}$$

because $\nabla \times (\nabla\psi) = 0$ for any ψ
$\to$ *gauge invariance*

Q: what condition needed to keep $\vec{E}$ unchanged?

Gauge Invariance

the physical electric field has

$$\vec{E} = -\nabla\phi - \frac{1}{c}\partial_t \vec{A} \tag{10}$$

and must remain the same when $\vec{A} \to \vec{A} + \nabla\psi$

but we have

$$\begin{aligned} \vec{E} \to \vec{E}' &= -\nabla\phi - \frac{1}{c}\partial_t \vec{A}' & (11) \\ &= -\nabla\left(\phi + \frac{1}{c}\partial_t\psi\right) - \frac{1}{c}\partial_t \vec{A} & (12) \end{aligned}$$

Q: and so?

$$\vec{E} \to \vec{E}' \;=\; -\nabla\left(\phi + \frac{1}{c}\partial_t\psi\right) - \frac{1}{c}\partial_t\vec{A} \tag{13}$$

and so to keep $\vec{E}' = \vec{E}$ requires

$$\phi \to \phi' = \phi - \frac{1}{c}\partial_t\psi \tag{14}$$

the $\vec{E}, \vec{B}$ preserving mappings

$$(\phi, \vec{A}) \to (\phi, \vec{A}) + (\partial_t\psi/c, \nabla\psi) \tag{15}$$

is a **gauge transformation**

a deep but also annoying property of electromagnetism
for our purposes, a useful but not unique choice

$$\nabla \cdot \vec{A} + \frac{1}{c}\partial_t\phi = 0 \tag{16}$$

"Lorentz gauge"

Maxwell Revisited

express Maxwell in terms of potentials: Coulomb

$$
\begin{aligned}
-\nabla \cdot \left(\nabla\phi - \frac{1}{c}\partial_t \vec{A} \right) &= -\nabla^2\phi - \frac{1}{c}\partial_t(\nabla \cdot \vec{A}) && (17) \\
&= 4\pi\rho_q && (18)
\end{aligned}
$$

and so in Lorentz gauge

$$\nabla^2\phi - \frac{1}{c^2}\partial_t^2\phi = -4\pi\rho_q \tag{19}$$

scalar potential satisfies a wave equation!
ϕ source is charge density ρ_q
changes in ϕ propagate at speed c

for *static* situation $\partial_t\phi = 0$, Poisson $\nabla^2\phi = -4\pi\rho_q$, and

∞

$$\phi(\vec{r}) = \int d^3\vec{r'} \, \frac{\rho_q(\vec{r'})}{|\vec{r'} - \vec{r}|} \tag{20}$$

Q: solution for full wave equation?

Scalar Potential and Retarded Time

general solution to

$$\nabla^2 \phi - \frac{1}{c^2} \partial_t^2 \phi = -4\pi \rho_q \tag{21}$$

turns out to be

$$\phi(\vec{r}, t) = \int d^3\vec{r}' \; \frac{\rho_q(\vec{r}', t')}{|\vec{r}' - \vec{r}|} = \int d^3\vec{r}' \; \left[\frac{\rho_q}{|\vec{r}' - \vec{r}|} \right]_{\text{ret}} \tag{22}$$

where source density $\rho_q(\vec{r}', t')$

is evaluated at **retarded time**

$$t' \equiv [t_{\text{ret}}] = t - \frac{|\vec{r} - \vec{r}'|}{c} \tag{23}$$

$\rightarrow \phi$ "learns" about changes in charge density at $\vec{r}'$

only after signal propagation time $ct_{\text{prop}} = |\vec{r}'|$

Maxwell and the Vector Potential

in terms of potentials, Ampère in Cartesian coords:

$$\nabla \times (\nabla \times \vec{A}) = \nabla^2 \vec{A} - \nabla(\nabla \cdot \vec{A}) \quad (24)$$

$$= \frac{4\pi}{c}\vec{j} + \frac{1}{c}\left(\nabla\phi + \partial_t \vec{A}\right) \quad (25)$$

so in Lorentz gauge

$$\nabla^2 \vec{A} - \frac{1}{c^2}\partial_t^2 \vec{A} = -\frac{4\pi}{c}\vec{j} \quad (26)$$

vector potential also satisfies a wave equation
source is current density $\vec{j}$

Q: solution?

each component A_i of vector potential satisfies

$$\nabla^2 A_i - \frac{1}{c^2}\partial_t^2 A_i = -\frac{4\pi}{c} j_i \tag{27}$$

formally identical to scalar potential equation
if we put $\phi \to A_i$ and $\rho_q \to j_i/c$

and thus we can import the solution:

$$A_i(\vec{r}, t) = \int d^3\vec{r}\,' \left[\frac{j_i}{|\vec{r}\,' - \vec{r}|}\right]_{\text{ret}} \tag{28}$$

$\to$ vector potential responds to current changes
after "retarded time" delay

Integral solutions for ϕ and $\vec{A}$ are huge!
Q: why? what's the Big Deal?

Recipe for Electromagnetic Fields

our mission: find $\vec{E}(\vec{r},t)$ and $\vec{B}(\vec{r},t)$
given charge $\rho_q(\vec{r},t)$ and current $\vec{j}(\vec{r},t)$ distributions

solution: first find potentials via

$$\phi(\vec{r},t) = \int d^3\vec{r'} \left[\frac{\rho_q}{|\vec{r'}-\vec{r}|}\right]_{\rm ret} \tag{29}$$

$$\vec{A}(\vec{r},t) = \int d^3\vec{r'} \left[\vec{j}|\vec{r'}-\vec{r}|\right]_{\rm ret} \tag{30}$$

from these, find fields via

$$\vec{E} = -\nabla\phi - \frac{1}{c}\partial_t\vec{A} \tag{31}$$

$$\vec{B} = \nabla \times \vec{A} \tag{32}$$

ta da!

in the 3-D spatial integrals

$$\phi(\vec{r},t) = -\int d^3\vec{r}' \ \left[\frac{\rho_q}{|\vec{r}' - \vec{r}|}\right]_{\text{ret}} \tag{33}$$

it is convenient (and pretty!) to recast as integrals over 4-D spacetime:

$$\phi(\vec{r},t) = -\int d^3\vec{r}' \ dt' \ \frac{\rho_q(\vec{r}', t')}{|\vec{r}' - \vec{r}|} \ \delta(t' - t + |\vec{r} - \vec{r}'|/c) \tag{34}$$

were the δ function enforces the retarded time condition

Q: What if charges are all pointlike?

Potentials from Point Charges

if N point charges, where ith charge q_i has trajectory with position $\vec{r}_i(t)$, and velocity $\vec{v}_i(t)$, then

$$\rho_q(\vec{r},t) = \sum_i q_i\ \delta^{(3)}(\vec{r}-\vec{r}_i) \quad (35)$$

$$\vec{j}(\vec{r},t) = \sum_i q_i\ v_i(t)\ \delta^{(3)}(\vec{r}-\vec{r}_i) \quad (36)$$

with Dirac δ-functions $\delta^{(3)}(\vec{r}-\vec{r}_i) = \delta(x-x_i)\ \delta(y-y_i)\ \delta(z-z_i)$

scalar potential due to *one charge* with $q_0, \vec{r}_0(t), \vec{v}_0(t)$ is

$$\phi(\vec{r},t) = q_0 \int d^3\vec{r}'\ dt'\ \frac{\delta^{(3)}(\vec{r}'-\vec{r}_0(t))}{|\vec{r}'-\vec{r}|}\ \delta(t'-t+|\vec{r}-\vec{r}'|/c) \quad (37)$$

space part of integral is easy

$$\phi(\vec{r},t) = q_0 \int dt'\ \frac{\delta\left(t'-t+|\vec{r}-\vec{r}_0(t')|/c\right)}{|\vec{r}-\vec{r}_0(t')|} \quad (38)$$

writing $\vec{R}(t') \equiv \vec{r} - \vec{r}_0(t')$
and $R(t') = |\vec{R}(t')|$, we have

$$\phi(\vec{r},t) = q_0 \int dt' \ \frac{\delta\left(t' - t + R(t')/c\right)}{R(t)} \tag{39}$$

and now the final δ function is nontrivial

math aside: fun properties of the δ function
$\delta(x)$ designed to give

$$\int f(y)\ \delta(y-x)\ dy = f(x) \tag{40}$$

but if δ argument is a function of the integration variable

$$\int f(y)\ \delta\left(g(x)\right)\ dy = \sum_{\text{roots}\,j} \frac{f(g(x_j))}{|dg/dx|_{x_j}} \tag{41}$$

where root x_j is the jth solution to $y - g(x) = 0$

here: define $t'' = t' - t + R(t')/c$
then $dt'' = dt' + \dot{R}(t')/c\ dt'$

Liénard-Wiechert Potentials

for point source with arbitrary trajectory, we have

$$\phi(\vec{r},t) = \frac{1}{1-\hat{n}\cdot\hat{\beta}_0(t_{\rm ret})}\frac{q_0}{R} \quad (42)$$

where $\hat{n} = \vec{r}/r$ and $\vec{\beta}_0(t) = \vec{v}_0(t)/c$

similarly, vector potential solution is

$$\vec{A}(\vec{r},t) = \frac{1}{1-\hat{r}\cdot\hat{\beta}_0(t_{\rm ret})}\frac{q_0\vec{v}_0(\vec{r},t_{\rm ret})}{R(t_{\rm ret})} \quad (43)$$

these are the **Liénard-Wiechert potentials**

Q: equipotential surfaces $\phi = const$ for stationary charge $\vec{r}_0(t) = const$?

Q: for charge with $\vec{v}_0$ large?

Q: implications?

potential factor $\kappa \equiv [1 - \hat{n} \cdot \hat{\beta}]_{\rm ret}$ is

- directional,
- velocity dependent, such that
- *potential* $\propto 1/\kappa$ *enhanced along direction of charge motion* and *potential suppressed opposite direction of charge motion*

$\Rightarrow$ expect forward "beaming" effects!

But we want the EM fields, not just potentials, so we need to evaluate

$$\vec{E} = -\nabla\phi - \frac{1}{c}\partial_t \vec{A} \quad (44)$$

$$\vec{B} = \nabla \times \vec{A} \quad (45)$$

using the beautiful Liénard-Wiechert point-source potentials where, $\phi = \phi[\vec{r}, t; \vec{r}_0(t), \vec{v}_0(t)]$ and $\vec{A} = \vec{A}[\vec{r}, t; \vec{r}_0(t), \vec{v}_0(t)]$

Q: what terms will appear in $\vec{E}$?

Electrodynamics of Moving Charges

after tedious algebra, we find:

$$\vec{E}(\vec{r},t) = q\left[\frac{(\hat{n}-\hat{\beta})(1-\beta^2)}{\kappa^2 R^2}\right]_{\text{ret}} + \frac{q}{c}\left[\frac{\hat{n}}{\kappa^3 R}\times\left\{(\hat{n}-\hat{\beta}) \ \times \dot{\vec{\beta}}\right\}\right]_{\text{ret}} \tag{46}$$

form is rich = complicated, but also complete and exact!
depends on charge position, velocity, and acceleration

magnetic field is

$$\vec{B}(\vec{r},t) = \left[\hat{n}\times\vec{E}(\vec{r},t)\right]_{\text{ret}} \tag{47}$$

Q: why did this have to be true?

Q: $\vec{E}$ result for charge at rest? with constant velocity?
Q: result at large R?

Electric "Velocity" Field

the first term = "velocity field"

$$\vec{E}(\vec{r},t)_{\text{vel}} = q\left[\frac{(\hat{n}-\vec{\beta})(1-\beta^2)}{\kappa^2 R^2}\right]_{\text{ret}} \tag{48}$$

- depends only on position and velocity evaluated at a *past* location of the particle
- velocity field *not isotropic* if particle moving

displacement from retarded position $\vec{r}_0(t_{\text{ret}})$
to the field position $\vec{r}$ is $\hat{n}c(t-t_{\text{ret}})$
to the current particle position $\beta c(t-t_{\text{ret}})$
so $\vec{E}$ *points to current* position!
$\rightarrow$ legal? yes! velocity constant, trajectory news always "available"