Astro 501: Radiative Processes
Lecture 12
Feb 11, 2013

Announcements:
- **Problem Set 4** due next Friday
- Happy Lunar New Year!

Last time: potentials, and the fields of moving charges
key idealized case: single point charge in arbitrary motion
Q: what does $\vec{E}$ *depend on?* $\vec{B}$*?*
Q: $\vec{E}$ *at constant velocity?*

Today: radiation by accelerated charges
feel it in your bones!

Electrodynamics of Moving Charges

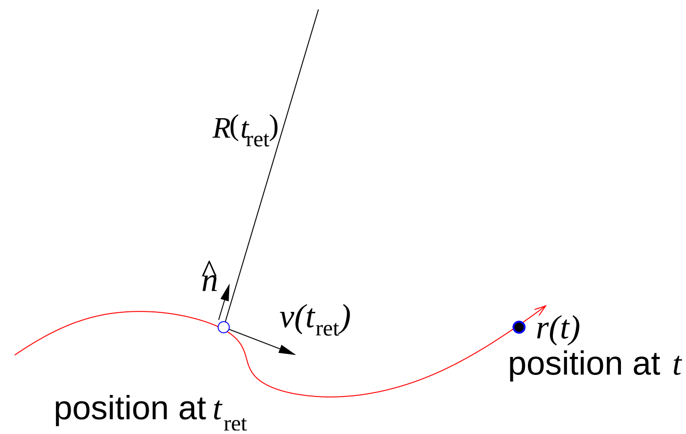
after tedious algebra, we find:

$$\vec{E}(\vec{r},t) = q\left[\frac{(\hat{n}-\vec{\beta})(1-\beta^2)}{\kappa^2 R^2}\right]_{\text{ret}} + \frac{q}{c}\left[\frac{\hat{n}}{\kappa^3 R}\times\left\{(\hat{n}-\vec{\beta})\ \times\dot{\vec{\beta}}\right\}\right]_{\text{ret}} \tag{1}$$

where $\kappa = 1 - \hat{n}\cdot\vec{\beta}$

magnetic field is

$$\vec{B}(\vec{r},t) = \left[\hat{n}\times\vec{E}(\vec{r},t)\right]_{\text{ret}}$$



the first term = “velocity field”

$\vec{E}$ *points to current* position!

$\longrightarrow$ legal? yes!

velocity constant, trajectory news always “available”

Electric Acceleration Field

electric velocity field $\propto 1/R^2$
but other *acceleration* term $\propto \dot{v}_0$

$$\vec{E}(\vec{r},t)_{\rm accel} = \frac{q}{c}\left[\frac{\hat{n}}{\kappa^3 R} \times \left\{(\hat{n}-\hat{\beta}) \ \times \dot{\hat{\beta}}\right\}\right]_{\rm ret} \quad (2)$$

drops with distance $\propto 1/R$: always larger at large R

for nonrelativistic motion, $\beta_0 = v_0/c \ll 1$,
and so to first order

$$\vec{E}(\vec{r},t)_{\rm accel} \approx \left[\frac{q}{c^2 R}\hat{n} \times (\hat{n} \times \vec{a})\right]_{\rm ret} \quad (3)$$

a huge result!

Q: if acceleration is linear, what is polarization?

at large distances

$$\vec{E}(\vec{r},t) \rightarrow \vec{E}(\vec{r},t)_{\text{accel}} \approx \left[\frac{q}{c^2 R}\hat{n} \times (\hat{n} \times \vec{a})\right]_{\text{ret}} \tag{4}$$

instantaneous $\vec{E}$ *direction* set by $\hat{a}$ and $\hat{n}$

if acceleration is linear $\rightarrow$ $\hat{a}$ fixed

then $\vec{E}$ lies within $(\hat{n},\hat{a})$ plane $\rightarrow$ *100% linearly polarized*

using $\vec{B} \rightarrow \hat{n} \times \vec{E}_{\text{accel}}$, the Poynting flux is

$$\vec{S} \approx \frac{c}{4\pi} E^2_{\text{accel}}\ \hat{n} = \frac{q^2}{4\pi c^3 R^2} \left|\hat{n} \times (\hat{n} \times \dot{\vec{\beta}})\right|^2 \hat{n} \tag{5}$$

Q: noteworthy features?

the Poynting flux is

$$\vec{S} \approx \frac{q^2}{4\pi c^3 R^2} \left| \hat{n} \times (\hat{n} \times \dot{\vec{\beta}}) \right|^2 \tag{6}$$

$S \propto R_{\rm ret}^{-2}$: *flux obeys inverse square law!*

Power per unit solid angle is

$$\frac{dP}{d\Omega} = R^2 \hat{n} \cdot \vec{S} \approx \frac{c}{4\pi} |R\vec{E}_{\rm accel}|^2 = \frac{q^2}{4\pi c^3} \left| \hat{n} \times (\hat{n} \times \dot{\vec{\beta}}) \right|^2 \tag{7}$$

independent of distance! *Q: why did this have to be true?*

Q: in which directions is $dP/d\Omega$ largest? smallest?
Q: radiation pattern?

Larmor Formula

Nonrelativistic charges radiate when accelerated!
Power per unit solid angle is

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c^3} \left| \hat{n} \times (\hat{n} \times \dot{\vec{\beta}}) \right|^2 \tag{8}$$

define angle Θ *between* $\vec{a}$ *and* $\hat{n}$ via $\hat{n} \cdot \hat{\beta} = \cos\Theta$:

$$\frac{dP}{d\Omega} = \frac{q^2 a^2}{4\pi c^3} \sin^2\Theta \tag{9}$$

a $\sin^2\Theta$ pattern!
$\rightarrow$ no radiation in direction of acceleration, maximum $\perp \vec{a}$

integrate over all solid angles: *total radiated power* is

$$P = \frac{q^2 a^2}{4\pi c^3} \int \sin^2\Theta \, d\Omega = \frac{2}{3} \frac{q^2}{c^3} a^2 \tag{10}$$

this will be our workhorse!
relates radiation to particle acceleration via $P \propto a^2$

Why does Acceleration Cause Radiation?

to get a physical intuition for why acceleration $\longrightarrow$ radiation
consider a particle rapidly *deceleraterated* from speed v to rest
over time δt

consider a later time $t \gg \delta t$

Q: field configuration near *particle* ($r \ll ct$) *?*

Q: field configuration near *particle* ($r \gg ct$)*?*

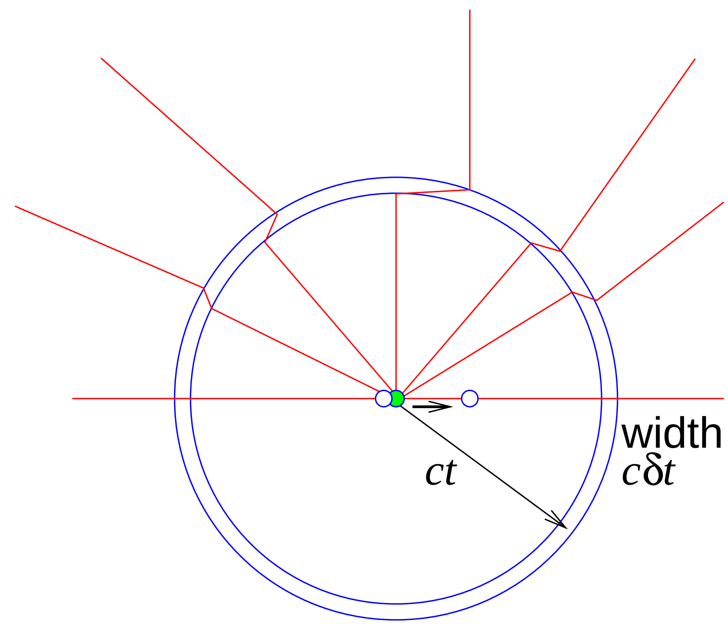
Q: consequences?

for fields track particle location expected for constant velocity

- nearby: $r \ll ct$, fields radial around particle at rest
- far away: $r \gg ct$: fields don't "know" particle has stopped
  $\rightarrow$ "anticipate" location displaced by ct from original particle
  radially oriented around this expected point

between the two regimes: $r = ct \pm c\delta t$
field lines must have "kinks" which

- have tangential field component
- tangential component is *anisotropic*
  and largest $\perp \vec{v}$



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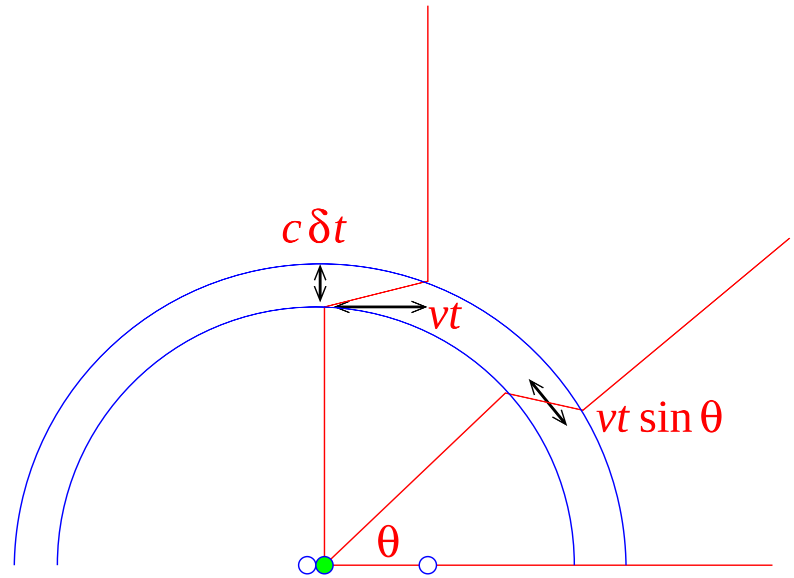
consider *vertical fieldline* $\perp \vec{v}$:

kink radial width $c\delta t$

kink tangential width $vt = (v/c)r$



tangential/radial ratio is $(v/\delta t)r/c^2$

but $v/\delta t = a$, average acceleration:

$\longrightarrow E_\perp/E_r = ar/c^2$

more generally, tangential width is

$vt \sin\Theta = (v/c)r \sin\Theta$

and so using Coulomb for E_r:

$$E_\perp = \frac{ar\sin\Theta}{c^2}E_r = \frac{qa}{c^2 r}\sin\Theta \tag{11}$$

and we recover Larmor:

$$\frac{dP}{d\Omega} = r^2 S = \frac{cr^2 E_\perp^2}{4\pi} = \frac{q^2 a^2}{4\pi c^3}\sin^2\Theta \tag{12}$$

Note: existence of kink and thus of radiation
demanded by combination of

- Gauss' law (field lines not created or destroyed in vacuum)
- finite propagation speed c

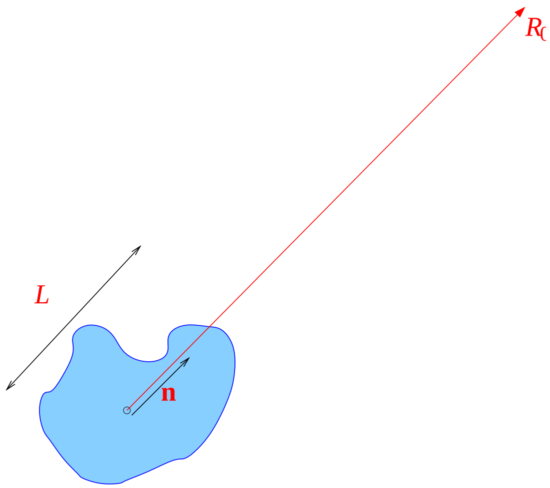
So far: field of a single point charge
Now: consider N particles, with q_i, $\vec{r}_i$, $\vec{u}_i = \dot{\vec{r}}_i$

Net $\vec{E}$ will be sum over all particles
Q: complications beyond "simple" bookkeeping?
Q: when will things simplify?

Approximate Phase Coherence

fields for each charge depend on it's *retarded time*
and these are different for each charge
$\longrightarrow$ leads to *phase differences* between particles
which we in general would have to track

When are phase differences not a problem?
When light-travel-time lags between particles
represent small phase differences



Let system size be L, and timescale for variations τ

if $\tau \gg L/c$, phase differences will be small

or: characteristic frequency is $\nu \sim 1/\tau$

so phase differences small if $c/\nu \gg L$, or $\lambda \gg L$

note that typical particle speeds $u \sim L/\tau$, so

phase coherence condition $\rightarrow u \ll c \rightarrow$ *nonrelativistic motion*

Dipole Approximation

so for **non-relativistic systems** we may ignore

- differences in time retardation, and
- the correction factor $\kappa = 1 - \hat{n} \cdot \vec{u}/c \rightarrow 1$

and thus we have

$$\vec{E}_{\mathrm{rad}} = \sum_i \frac{q_i}{c^2} \frac{\hat{n} \times (\hat{n} \times \vec{a}_i)}{R_i} \tag{13}$$

but the system has $R_i \approx R_0 \gg L$, and so

$$\vec{E}_{\mathrm{rad}} = \hat{n} \times \left(\frac{\hat{n}}{c^2 R_0} \times \sum_i q_i \vec{a}_i \right) = \frac{\hat{n} \times (\hat{n} \times \ddot{\vec{d}})}{c^2 R_0} \tag{14}$$

where the **dipole moment** is

$$\vec{d} = \sum_i q_i \vec{r}_i \tag{15}$$

for a non-relativistic dipole, we have

$$\vec{E}_{\text{rad}} = \frac{\hat{n} \times (\hat{n} \times \ddot{\vec{d}})}{c^2 R_0} \tag{16}$$

this *dipole approximation* gives: power per unit solid angle

$$\frac{dP}{d\Omega} = \frac{\ddot{d}^2}{4\pi c^3} \sin^2 \Theta \tag{17}$$

and the total power radiated

$$\frac{dP}{d\Omega} = \frac{2}{3} \frac{\ddot{d}^2}{c^3} \tag{18}$$

consider a dipole that maintains the same orientation $\vec{d}$

$$E(t) = \ddot{d}(t) \frac{\sin \Theta}{c^2 R_0} \tag{19}$$

using Fourier transform of $d(t)$, we have

$$d(t) = \int e^{-i\omega t} \tilde{d}(\omega) \ d\omega \tag{20}$$

and so

$$\tilde{E}(\omega) = -\omega^2 \tilde{d}(\omega) \frac{\sin \Theta}{c^2 R_0} \tag{21}$$

and thus the energy per solid angle and frequency is

$$\frac{dW}{d\Omega d\omega} = \frac{1}{c^3} \omega^4 \ \left| \tilde{d}(\omega) \right|^2 \sin^2 \Theta \tag{22}$$

and

$$\frac{dW}{d\omega} = \frac{8\pi}{3c^3} \omega^4 \ \left| \tilde{d}(\omega) \right|^2 \tag{23}$$

- note the $\omega^4 \propto \lambda^{-4}$ dependence
- and $\tilde{d}(\omega)$: *dipole frequencies control radiation frequencies*