Astro 501: Radiative Processes
Lecture 15
Feb 18, 2013

Announcements:

- **Problem Set 5** due 5pm Friday

Last time: bremsstrahlung

*Q: what is it?*

*Q: what interactions, trajectories are relevant?*

*Q: what does bremsstrahlung emission $j_\nu$ depend on?*

Bremsstrahlung = "breaking radiation"
= radiation from decelerated charge particles

electron and ion scattered by same Coulomb force (Newton III)
But $a_{\rm i}/a_e = m_e/m_{\rm i} < 10^{-3} \to$ ion acceleration negligible
$\to$ focus electron acceleration in static field of ion

Our order-of-magnitude estimate for the emission coefficient from non-relativistic bremsstrahlung:

$$j_\nu \sim \frac{e^6 Z^2}{m_e c^3 v} n_e n_{\rm i} \tag{1}$$

*Q: what's the basic physical picture?*

*Q: notable features? what didn't we get from order of mag?*

*Q: how can we do the classical calculation more carefully?*

# Bremsstrahlung: Physical Picture

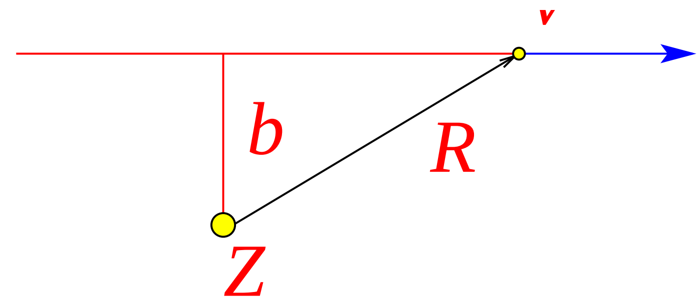
we are interested in the motion of an electron through a plasma

we approximate this as a series of
- *two-body electron-ion* scattering events
- *unbound Coulomb* trajectories: *hyperbolæ*
  $\rightarrow$ asymptotically free, scattered through small angle
- acceleration maximum at closest approach $b$
  lasting for scattering time $\tau = b/v$
- burst of radiation over this time, frequency $\nu \sim 1/\tau$

So net effect is
- many scattering events
- a series of small-angle scatterings
- and radiation bursts at different frequencies

# Bremsstrahlung: Classical Calculation

Consider electron with initial speed $v$
with *impact parameter* $b$
moving fast enough so that
*scattered through small angle*




dipole moment $\vec{d} = -e\vec{R}$, with second derivative

$$\ddot{\vec{d}} = -e\dot{\vec{v}} \tag{2}$$

take Fourier transform

$$-\omega^2 \ddot{\vec{d}} = -\frac{e}{2\pi}\int_{-\infty}^{\infty} \dot{\vec{v}} e^{i\omega t} \, dt \tag{3}$$

where: $\vec{v}(t)$ is *an unbound Coulomb trajectory*:
$\rightarrow$ *hyperbola* in space, complicated function of time
but: $\dot{\vec{v}}(\omega)$ simplifies in limiting cases
$\rightarrow$ compare $\omega$ and collision time $\tau = b/v$
*Q:* $\omega\tau \gg 1$? $\omega\tau \ll 1$?

$$-\omega^2 \,\ddot{\vec{d}} = -\frac{e}{2\pi}\int_{-\infty}^{\infty} \dot{\vec{v}} e^{i\omega t}\, dt \tag{4}$$

but $\vec{v}(t)$ only changes on timescale $\tau$:
for $\omega\tau \gg 1$, many oscillations during acceleration
complex phase averages out: $\vec{v}(\omega) \to 0$

for $\omega\tau \ll$, complex exponent unchanged during accel
phase unimportant: $\vec{v}(\omega) \to \int \dot{\vec{v}}\ dt = \Delta\vec{v}$

and thus the dipole moment has

$$\vec{d}(\omega) \to \begin{cases} \frac{e}{2\pi\omega^2}\Delta\vec{v} & \omega\tau \ll 1 \\ 0 & \omega\tau \gg 1 \end{cases} \tag{5}$$

Energy emitted per unit frequency

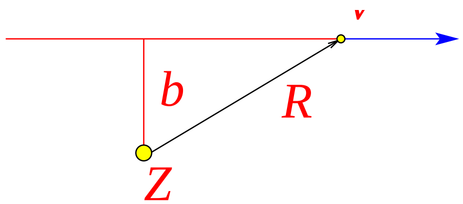
$$\frac{dW}{d\omega} \rightarrow \begin{cases} \frac{2e^2}{3\pi c^3}|\Delta\vec{v}|^2 & \omega\tau \ll 1 \\ 0 & \omega\tau \gg 1 \end{cases} \tag{6}$$

Now find $\Delta\vec{v}$: for small deflection

$$\Delta v \approx \Delta v_\perp = \int F_z \, dt \tag{7}$$

$$= \frac{Ze^2}{m_e} \int \frac{b}{(b^2 + v^2 t^2)^{3/2}} dt \tag{8}$$

$$= \frac{2Ze^2}{m_e b v} \tag{9}$$



energy emitted per electron

$$\frac{dW(b)}{d\omega} = \begin{cases} \frac{8Z^2e^2}{3\pi c^3 M_e^2 v b^2} & \omega\tau \ll 1 \\ 0 & \omega\tau \gg 1 \end{cases} \tag{10}$$

power emitted power per volume

$$\frac{dW(b)}{dV\ d\omega\ dt} = n_e \frac{dW}{d\omega}\frac{d\mathcal{N}_{\rm i}}{dt} = 2\pi n_e n_{\rm i} \int_{b_{\rm min}}^{b_{\rm max}} \frac{dW(b)}{d\omega}\ b\ db \tag{11}$$

approximate with low-frequency result:

$$q_\nu = 4\pi j_\nu = \frac{dW}{dV\ d\omega\ dt} = \frac{16Z^2e^2}{3\pi c^3 m_e^2 v} n_e n_{\rm i}\ \ln\left(\frac{b_{\rm max}}{b_{\rm min}}\right) \tag{12}$$

compare/contrast with order-of-magnitude:

- linear scaling with $e$ and ion density
- $1/v$ scaling
- independence of $b$ range $\to$ log dependence
- independence with $\nu, \omega$: "flat" emission spectrum

# Impact Parameter Range

bremsstrahlung emission at speed $v$, frequency $\omega$
depends *logarithmically* on the limits
$b_{\rm min}, b_{\rm max}$ of impact parameter
within our classical, small-angle-scattering treatment

*lower limit*

- quantum mechanics: $\Delta x \ \Delta p \gtrsim \hbar$
  $\rightarrow b_{\rm min}^{(1)} > h/mv$
- small-angle: $\Delta v/v \sim Ze^2/bmv^2 < 1$
  $\rightarrow b_{\rm min}^{(2)} > Ze^2/mv^2$

*upper limit*

for a fixed $\omega$ and $v$, max impact parameter is $b_{\rm max} \sim v/\omega$

∞

fortunately: log dependence on limits
$\rightarrow$ results not very sensitive to details of choices

# Single-Velocity Bremsstrahlung

convenient, conventional form for bremsstrahlung emission also known as **free-free** emission

$$4\pi \ j_\omega(\omega, v) = \frac{16\pi}{3\sqrt{3}} \frac{Z^2 e^6}{m_e^2 c^3 v} \ n_{\rm i} n_e \ g_{\rm ff}(\omega, v) \quad (13)$$

uses the dimensionless correction factor or **Gaunt factor**

$$g_{\rm ff}(\omega, v) = \frac{\sqrt{3}}{\pi} \ \ln\left(\frac{b_{\rm max}}{b_{\rm min}}\right) \quad (14)$$

- accounts for log factor
- typically $g_{\rm ff} \sim 1$ to few
- tables and plots available

# Thermal Bremsstrahlung

so far: calculated bremsstrahlung emission for
a *single electron velocity* $v$
$\rightarrow$ a "beam" of mono-energetic electrons

but in real astrophysical applications
there is a *distribution* of electron velocities
usually: a *thermal* distribution

so we wish to find
the *mean* or *expected* emission $\langle j_{\nu,\mathrm{brem}} \rangle$
for a thermal distribution of velocities

*Q: order-of-magnitude expectation?*

# Thermal Bremsstrahlung: Order-of-Magnitude

order-of-magnitude emission for single $v$:

$$j_\nu \sim \frac{e^6 Z^2}{m_e c^3 v} n_e n_i \tag{15}$$

i.e., $j_\nu \sim 1/v$

thus, thermal average

$$\langle j_\nu \rangle \sim \frac{e^6 Z^2}{m_e c^3 v_T} n_e n_i \tag{16}$$

with $v_T$ a typical thermal velocity

find $v_T$ from equipartition: $m_e v_T^2 \sim kT \rightarrow v_T \sim \sqrt{kT/m_e}$

*Q: how do we approach the honest, detailed calculation?*

*Q: yet more new formalism?*

# Thermal Particles: Non-Relativistic Limit

recall: semiclassically, particle behavior in *phase space* $(\vec{x}, \vec{p})$
described by *distribution function* $f$:

- Heisenberg: minimum phase-space "cell" size $dx\ dp = h$
- particle number $dN = g/h^3\ f(\vec{x}, \vec{p})\ d^3\vec{x}\ d^3\vec{p}$

a *dilute*=non-degenerate, *non-relativistic* particle species
of mass $m$ at temperature $T$
has distribution function

$$f_{\text{therm}}(p) \propto e^{-p^2/2mT} \tag{17}$$

and thus has number density $n \propto \int e^{-p^2/2m_eT} d^3\vec{p} \propto \int e^{-m_e v^2/2kT} d^3\vec{v}$

*Q: how to compute thermal averaged bremsstrahlung emission?*

Bremsstrahlung emissivity depends on electron properties via

$$j_\nu(\nu, T) = \langle j_\nu(\nu, v) \rangle \propto \left\langle \frac{g_{\rm ff}(\nu, v)\ n_e}{v} \right\rangle \tag{18}$$

where

$$\left\langle \frac{g_{\rm ff}(\omega, v)\ n_e}{v} \right\rangle \sim \int_{v_{\rm min}}^{\infty} \frac{g_{\rm ff}(\omega, v)}{v}\ e^{-m_e v^2/2kT}\ d^3\vec{v} \tag{19}$$

Note lower limit $v_{\rm min}$ at fixed $\nu$
$\rightarrow$ minimum electron velocity needed to radiate photon of energy $\nu$

*Q: what value should this have? effect on final result?*

energy conservation: to make photon of frequency $\nu$
electron needs kinetic energy $m_e v^2/2 > h\nu$, so

$$v_{\rm min} = \sqrt{\frac{2h\nu}{m_e}} \tag{20}$$

thus exponential factor has

$$e^{-\frac{m_e v^2}{2kT}} = e^{-\frac{m_e v_{\rm min}^2}{2kT}} \; e^{-\frac{m_e (v^2 - v_{\rm min}^2)}{2kT}} = e^{-\frac{h\nu}{kT}} \; e^{-\frac{m_e (v^2 - v_{\rm min}^2)}{2kT}}$$

$\rightarrow$ overall factor $e^{-h\nu/kT}$ in thermal average
$\rightarrow$ photon production thermally suppressed at $h\nu > kT$

thermal bremsstrahlung = "free-free" emission result:

$$4\pi j_{\nu,\rm ff}(T) = \frac{2^5 \pi \; Z^2 \; e^6}{3 \; m_e c^3} \left( \frac{2\pi}{3 m_e kT} \right)^{1/2} \; \bar{g}_{\rm ff}(\nu, T) \; e^{-h\nu/kT} \; n_e \; n_{\rm i} \tag{21}$$

with $\bar{g}_{\rm ff}(\nu, T)$ the *velocity-averaged thermal Gaunt factor*
*Q: spectral shape for optically thin plasma? implications?*
*Q: integrated emission?*

$$4\pi j_{\nu,\mathrm{ff}}(T) = \frac{2^5\pi\ Z^2\ e^6}{3\ m_e c^3}\left(\frac{2\pi}{3m_e kT}\right)^{1/2}\ \bar{g}_{\mathrm{ff}}(\nu,T)\ e^{-h\nu/kT}\ n_e\ n_{\mathrm{i}} \quad (22)$$

main frequency dependence is $j_\nu \propto e^{-h\nu/kT}$
$\rightarrow$ flat spectrum, cut off at $\nu \sim kT/h$
$\rightarrow$ can use to determine temperature of hot plasma (PS5)

integrated bremsstrahlung emission:

$$4\pi j_{\mathrm{ff}}(T) = 4\pi \int j_{\nu,\mathrm{ff}}(T)\ d\nu \quad (23)$$

$$= \frac{2^5\pi\ Z^2\ e^6}{3\ m_e c^3}\left(\frac{2\pi kT}{3m_e}\right)^{1/2}\ \bar{g}_{\mathrm{B}}(T)\ e^{-h\nu/kT}\ n_e\ n_{\mathrm{i}} \quad (24)$$

$$= 1.4 \times 10^{-27}\ \mathrm{erg\ s^{-1}\ cm^{-3}}\ \bar{g}_{\mathrm{B}}\ \left(\frac{T}{\mathrm{K}}\right)^{\frac{1}{2}}\ \left(\frac{n_e}{1\ \mathrm{cm^{-3}}}\right)\ \left(\frac{n_{\mathrm{i}}}{1\ \mathrm{cm^{-3}}}\right)$$

with $\bar{g}_{\mathrm{B}}(T) \sim 1.2 \pm 0.2$ a frequency-averaged Gaunt factor
*Q: all of this was for emission–what about the thermal bremsstrahlung absorption coefficient?*

# Thermal Bremsstrahlung Absorption

for *thermal* system, *Kirchoff's law:* $S_\nu = B_\nu(T) = j_\nu/\alpha_\nu$

thus we have

$$\alpha_{\nu,\mathrm{ff}} = \frac{j_{\nu,\mathrm{ff}}}{B_\nu(T)} = \frac{4\ Z^2\ e^6}{3\ m_e h c}\left(\frac{2\pi}{3 m_e k T}\right)^{1/2} \bar{g}_{\mathrm{ff}}(\nu, T)\ \nu^{-3}\ \left(1 - e^{-h\nu/kT}\right) n_e\, n_{\mathrm{i}}$$

limits:

- $h\nu \gg kT$: $\alpha_{\nu,\mathrm{ff}} \propto \nu^{-3}$
- $h\nu \ll kT$: $\alpha_{\nu,\mathrm{ff}} \propto \nu^{-2}$

*Q: sketch optical depth vs $\nu$? implications?*

# Bremsstrahlung Self-Absorption

bremsstrahlung optical depth at small $\nu$:

$$\tau_\nu \propto \alpha_{\nu,\mathrm{ff}} \propto \nu^{-3} \tag{25}$$

*guaranteed optically thick* below some $\nu$
$\rightarrow$ free-free emission is absorbed inside plasma:
**bremsstrahlung self-absorption**

thus observed plasma spectra should have three regimes

- small $\nu$: $\tau_\nu \gg 1$, optically thick, $I_\nu \rightarrow B_\nu \propto \nu^3$
- $h\nu < kT$: optically thin, $I_\nu \rightarrow j_\nu s$ *flat* vs $\nu$
- $h\nu \gg kT$: thermally suppressed, $I_\nu \rightarrow j_\nu s \sim e^{-h\nu/kT}$



*Q: expected X-ray count spectrum for supernova remnant?*

`www: observations`