Astro 501: Radiative Processes
Lecture 17
Feb 22, 2013

Announcements:

- **Problem Set 5** due 5pm today
- good news: *no problem set next Friday!*
- bad news: **Midterm in class next Friday**
  info online

Last Time: Special Relativity and Radiation

Q: when is special relativity astrophysically important?

Q: angular radiation pattern from a relativistic emitter?

Today: other key Special Relativity results

mostly results: derivation, formalism in R&L, Jackson, etc.

Relativistic Beaming

for light emitted in K' at $\theta' = \pi/2$
observed angle after boosting is

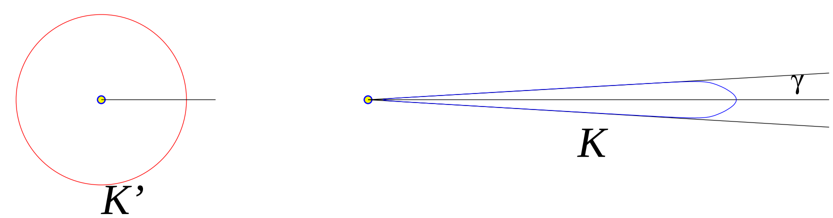
$$\tan\theta = \frac{1}{\gamma v/c} \tag{1}$$

and thus

$$\sin\theta = \frac{1}{\gamma} \tag{2}$$

if emitted K' is highly relativistic,
then $\gamma \gg 1$, and

$$\theta \rightarrow \frac{1}{\gamma}$$

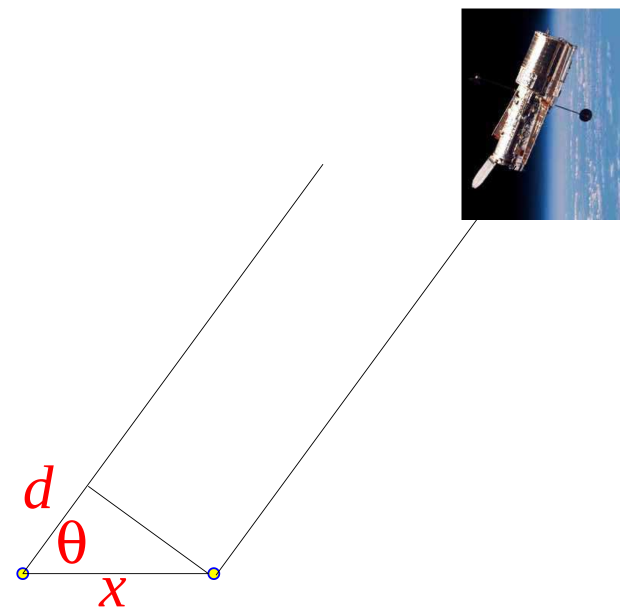


i.e., a small forward angle!
a highly relativistic emitter gives a **beamed radiation pattern**
strongly concentrated ahead of emitter direction

Relativistic Doppler Effect

emitter moves with speed v wrt observer

in emitter frame K':
light has (rest) frequency ω'
first wave crest emitted at $t' = 0$
second wave crest emitted at $t' = 2\pi\omega'$



in observer frame K:
observe light at angle θ
second wave crest after emitter travels distance $x = vt$

difference in observed light arrival times is

ω

$$\delta t = t - d/c = (1 - v \cos\theta/c)t \tag{3}$$

difference in observed light arrival times is

$$\delta t = t - d/c = (1 - v\cos\theta/c)t \tag{4}$$

and since $t = t'/\gamma$, we have

$$\omega' = \left(1 - \frac{v}{c}\cos\theta\right)\frac{\omega}{\gamma} \tag{5}$$

so: light emitted at rest frequency $\omega' = \omega_{\rm emit}$
is observed at angle θ to have frequency

$$\omega = \left(1 - \frac{v}{c}\cos\theta\right)\frac{\omega'}{\gamma} \tag{6}$$

and thus

$$\omega_{\rm obs} = \gamma\left(1 - \frac{v}{c}\cos\theta\right)\omega_{\rm emit} \tag{7}$$

relativistic Doppler formula

Four Vectors

recall: in 3-D space, isotropy $\rightarrow$ rotational invariance
i.e., experiments give same result if rotate entire apparatus through some angle

laws of physics must display rotational invariance
$\rightarrow$ most conveniently done by (3-D) vector notation, e.g.,
vector statement $\vec{F} = m\vec{a}$ automatically rotationally invariant
$\rightarrow$ in a rotated from $\vec{F}' = m\vec{a}'$: same law

take a similar approach to 4-D Lorentz transformation
define coordinate **4-vector**

$$X = (x_0, x_1, x_2, x_3) = (ct, x, y, z) = (ct, \vec{r}) \tag{8}$$

Four Vectors

- Lorentz transformation to frame with $\vec{v} = v\hat{x}$

$$X = \begin{pmatrix} x'_0 \\ x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{pmatrix} \gamma(x_0 - \beta x_1) \\ \gamma(x_1 - \beta x_0) \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

- dot product

$$X \cdot Y = -x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3 = -x_0 y_0 + \vec{x} \cdot \vec{y} \quad (9)$$

$$= \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix} \quad (10)$$

gorgeous feature: *dot product result same in any Lorentz frame*

i.e., $X \cdot Y = X' \cdot Y'$

$\Rightarrow$ *dot product is Lorentz invariant! everyone agrees!*

invariance of dot product very useful
you pick convenient frame, evaluate dot product
$\rightarrow$ result good in any frame

thus everyone agrees on 4-vector "norm" like $X^2 \equiv X \cdot X$
Q: what is this quantity? what is the physical significance?

Q: what does it mean physically when $X \cdot X = 0$*?* > 0*?* < 0*?*

note: different conventions for dot product sign
i.e., sign of Minkowski metric
sometimes $X \cdot Y = x_0 y_0 - \vec{x} \cdot \vec{y}$

Relativistic Velocity

Recall: for events separated by (dt, dx, dy, dz)
invariant interval$/c$ = **proper time**

$$d\tau^2 = \frac{ds^2}{c^2} = dt^2 - d\vec{r}^2/c^2 = dt^2\ (1 - u^2/c^2) \tag{11}$$

$$d\tau = \frac{dt}{\gamma} \tag{12}$$

where $\vec{u} = d\vec{r}/dt$, and $\gamma_u = (1 - v^2/c^2)^{-1/2}$
$\to d\tau =$elapsed time in rest frame of the event pair

Take proper time derivative of position 4-vector
$\to$ defines **4-velocity**: for $\mu \in 0, 1, 2, 3$

$$U^\mu = \frac{dx^\mu}{d\tau} \tag{13}$$

∞

Q: $\mu = 0$ component? other components?

$$U^\mu = \frac{dx^\mu}{d\tau} \tag{14}$$

zero (i.e., time) component

$$U^0 = \frac{c\ dt}{d\tau} = c\gamma_u \frac{dt}{dt} = c\gamma_u \tag{15}$$

space components $i \in 1, 2, 3$

$$U^i = \frac{dx^i}{d\tau} = \gamma_u \frac{dx^i}{dt} = \gamma_u\ u^i \tag{16}$$

and thus $U = (\gamma c, \gamma \vec{u})$

- Lorentz transformation properties: *same as space 4-vectors*
- norm:

$$U^2 = -c^2 \frac{dt^2}{d\tau}{}^2 + \frac{d\vec{r}^2}{d\tau}{}^2 = -c^2 \frac{d\tau^2}{d\tau^2} = -c^2 \tag{17}$$

Energy and Momentum

for a particle of (rest) mass m, define **4-momentum**

$$P^\mu = mU^\mu = \frac{dx^\mu}{d\tau} \tag{18}$$

- zero component: **relativistic energy**

$$P^0 = mU^0 = mc\gamma_u = \frac{E}{c} \tag{19}$$

in $\vec{u} = 0$ rest frame, $E = mc^2$ *rest energy*

- space components: **relativistic momentum**

$$P^i = mU^i = \gamma_u \; m \; u^i \tag{20}$$

Q: norm? what if massless?

• norm: $c^2P^2 = m^2c^2U^2 = -E^2 + (cp)^2 = -m^2c^4$
$\rightarrow E = \sqrt{(c\vec{p})^2 + (mc^2)^2}$

• relativistic limit: for $u/c \rightarrow 1$
$P \rightarrow (\gamma mc, \gamma mc)$: $E \approx p$

• Energy and momentum conservation all in one
if no forces act $P_{\text{init}} = P_{\text{final}}$

• massless particles?
recall: $ds^2 = c^2 d\tau^2 = cdt^2 - d\vec{r}^2 = 0$
$\rightarrow$ can't take proper time derivative
$\rightarrow$ but still can define 4-momentum $P = (E/c, \vec{p})$
with $P^2 = 0 \rightarrow E = cp$

Electromagnetic Fields in Relativity

Electric and magnetic fields are vectors in 3-D space
how to characterize in 4-D?

Turns out: natural to define **field tensor**

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}$$

Lorentz transformation of fields?
if boost with velocity $\vec{v} = c\vec{\beta}$

$$E'_{\parallel} = E_{\parallel} \quad (21)$$

$$B'_{\parallel} = B_{\parallel} \quad (22)$$

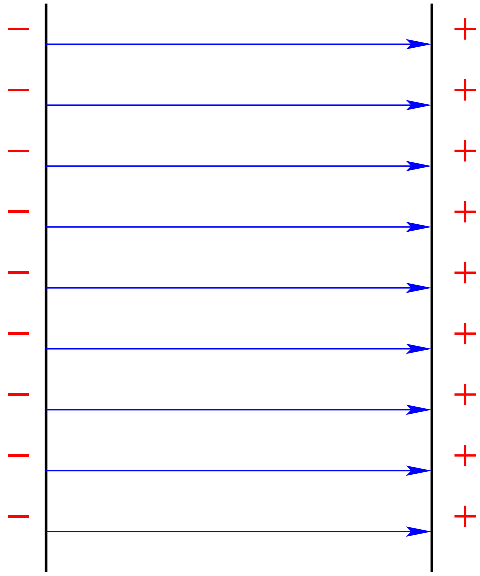
$$E'_{\perp} = \gamma(E_{\perp} + \vec{\beta} \times \vec{B}) \quad (23)$$

$$B'_{\perp} = \gamma(B_{\perp} - \vec{\beta} \times \vec{E}) \quad (24)$$

Field Transformation: Intuitive Picture

consider a charged *capacitor*

plates have *surface charge density* σ



from Gauss' law, electric field has

$$\int \vec{E} \cdot d\vec{S} = ES = 4\pi Q_{\text{enc}} = 4\pi\sigma S \quad (25)$$

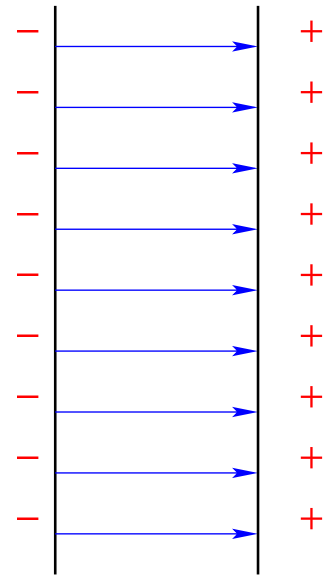
and so $E = 4\pi\sigma$

Q: what if we boost along $\vec{E}$*?*

Q: what if we boost perpendicular *to* $\vec{E}$*?*

boosting *along* $\vec{E}$: length contraction

$\rightarrow$ plates appear closer



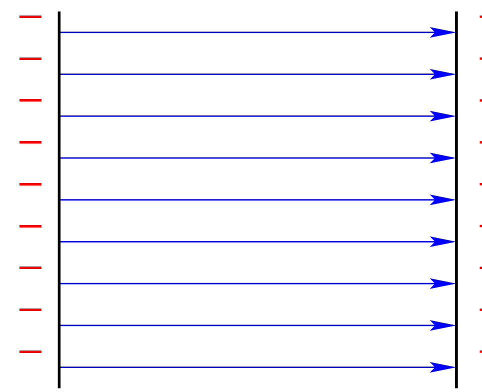
but surface charge density *unchanged*

$\rightarrow$ electric field

$E' = E'_{\parallel} = 4\pi\sigma = E$ *unchanged*

boosting *orthogonal* to $\vec{E}$:

length contraction $\rightarrow$ plates "squished"



charge conserved

$Q'_{\rm enc} = Q_{\rm enc} = 4\pi S'\sigma'$

but contracted surface has area $S' = S/\gamma$,

so surface charge density *increases*: $\sigma' = \gamma\sigma$

$\rightarrow$ electric field $E' = E'_{\perp} = 4\pi\sigma' = \gamma E$ *enhanced*

Electromagnetic Forces

define 4-acceleration $a_\mu = d^2x^\mu/d\tau^2 = dU^\mu/d\tau$

Relativistic version of force law:

$$ma^\mu = \frac{dP_\mu}{d\tau} = qU_\nu F_\mu^\nu \tag{26}$$

zero component: energy conservation

$$\frac{dE}{dt} = q\vec{v} \cdot \vec{E} \tag{27}$$

space components: Lorentz force

$$\frac{d\vec{p}}{dt} = q\vec{E} + q\vec{v} \times \vec{B} \tag{28}$$

Emission from a Relativistic Accelerated Charge

Consider a charge q accelerating relative to an observer
useful to go to charge's *instantaneous rest frame* K'
$\rightarrow$ in K', emission is given by nonrelativistic Larmor!

in time dt', emitted energy is dW'
but *net momentum of emitted radiation* is $d\vec{p}' = 0$
since Larmor (dipole) emission is front-back symmetric

in observer frame (velocity $-v$):
energy $dW = \gamma dW'$ emitted in time $dt = \gamma dt'$, so:

$$P = \frac{dW}{dt} = \frac{dW'}{dt'} = P' \tag{29}$$

total power emitted is Lorentz invariant!

Larmor: $P' = 2q^2/3c^3\ |\vec{a}'|^2$
want to re-express using 4-acceleration

can show: in instantaneous rest frame, $a^{0\prime} = 0$
and thus $|\vec{a}'|^2 = a \cdot a$
Lorentz-invariant Larmor expression for total radiated power

$$P = \frac{2}{3}\frac{q^2}{c^3} a \cdot a \tag{30}$$

manifestly invariant, can evaluate in any frame

$$P = \frac{2}{3}\frac{q^2}{c^3} a \cdot a \tag{31}$$

in instantaneous rest frame, 4-acceleration transforms as

$$\begin{aligned} a'_{\parallel} &= \gamma^3 a_{\parallel} && (32) \\ a'_{\perp} &= \gamma^2 a_{\perp} && (33) \\ & && (34) \end{aligned}$$

and so power emitted is

$$\begin{aligned} P &= \frac{2}{3}\frac{q^2}{c^3} \vec{a}' \cdot \vec{a}' = \frac{2}{3}\frac{q^2}{c^3} (a'^2_{\perp} + a'^2_{\parallel}) && (35) \\ &= \frac{2}{3}\frac{q^2}{c^3}\, \gamma^4 \, (a^2_{\perp} + \gamma^2 a^2_{\parallel}) && (36) \end{aligned}$$