Astro 501: Radiative Processes
Lecture 19
Feb 27, 2013

Announcements:

- good news: *no problem set this week!*
- bad news: **Midterm in class next time**
  info online

Last time: cosmic rays

Q: what are cosmic rays?

Q: what is their energy spectrum?

Q: where are cosmic rays found?

cosmic ray spectrum clearly **nonthermal**
rather: a succession of *power laws*

- observed CR electrons: roughly

$$\mathcal{I}_E(e) \propto E^{-3} \qquad (1)$$

and thus $I_E(e) = E/\mathcal{I}_E(e) \propto E^{-2}$

- protons w/ $1 \text{ GeV} \lesssim E \lesssim 300 \text{ TeV}$:

$$I_E(p) \simeq 1.4 \left(\frac{E}{\text{GeV}}\right)^{-s} \text{ protons cm}^{-2}\text{ s}^{-1}\text{ sr}^{-1}\text{GeV}^{-1} \qquad (2)$$

  where spectral index (“slope”) $s \simeq 2.7$
- beyond “knee’ at $E_{\text{knee}} \sim 10^{15}$ eV
  power law index steepens to $s \sim 3$
- then beyond “ankle” at $E_{\text{anlke}} \simeq 10^{18}$ eV, flattens again

Synchrotron Radiation

Relativistic Motion in a Uniform B Field

Consider a relativistic classical particle, mass m, charge q
moving in a uniform magnetic field $\vec{B}$
with no electric field

Equations of motion: relativistic *energy*

$$\frac{dW}{dt} = mc^2 \frac{d\gamma}{dt} = q\ \vec{v} \cdot \vec{E} = 0 \tag{3}$$

and so γ is *constant* and hence $|\vec{v}|$ is too

Equations of motion: *momentum*

$$\frac{d\vec{p}}{dt} = m \frac{d}{dt} \gamma \vec{v} = \frac{q}{c} \vec{v} \times \vec{B} \tag{4}$$

$$\frac{d}{dt}\gamma\vec{v} = \frac{q}{mc}\vec{v} \times \vec{B} \tag{5}$$

but γ and $|\vec{v}|$ are constant, so

$$\frac{d}{dt}\vec{v} = \frac{q}{\gamma mc}\vec{v} \times \vec{B} \tag{6}$$

take dot product with $\vec{B}$

$$\vec{B} \cdot \frac{d}{dt}\vec{v} = B\frac{d}{dt}v_{\parallel} = 0 \tag{7}$$

$\rightarrow$ velocity component $v_{\parallel}$ parallel to $\vec{B}$ is constant

decompose velocity into $\vec{v} = \vec{v}_{\parallel} + \vec{v}_{\perp}$

$$\frac{d}{dt}\vec{v}_{\perp} = \frac{q}{\gamma mc}\vec{v}_{\perp} \times \vec{B} \tag{8}$$

Q: resulting motion?

$$\frac{d}{dt}\vec{v}_\perp = \frac{q}{\gamma mc}\vec{v}_\perp \times \vec{B} = \vec{v}_\perp \times \vec{\omega}_B \tag{9}$$

perpendicular velocity *precesses* around $\vec{B}$
with gyrofrequency

$$\vec{\omega}_B = \frac{q}{\gamma mc}\vec{B} \tag{10}$$

note: nonrelativistic gyrofrequency $\omega_{B,\mathrm{nr}} = qB/mc$
is independent of v
but in relativistic case has factor $1/\gamma$

Q: resulting motion of charge?

orthogonal to $\vec{B}$, particle with speed $v_\perp$ moves in circle with **gyroradius**

$$r_{\rm g} = \frac{v_\perp}{\omega_B} = \frac{mc\gamma v_\perp}{qB} = \frac{cp_\perp}{qB} \tag{11}$$

thus the general motion is a combination of

- constant velocity $v_\parallel$ along $\vec{B}$
- uniform circular motion in plane orthogonal $\vec{B}$

net result: **spiral** around $\vec{B}$

numerically: gyroradius

$$r_{\rm g} = 3.3 \times 10^{12}\ \text{cm} \left(\frac{cp_\perp}{1\ \text{GeV}}\right) \left(\frac{1\ \mu\text{Gauss}}{B}\right) \tag{12}$$

Q: why these choices for $p_\perp$ and B? implications?
Q: what if p very very large?

typical "blue collar" cosmic ray energy $E \sim cp \sim 1\text{GeV}$
and typical interstellar magnetic field $B \sim 1\ \mu\text{Gauss}$

thus typical cosmic ray gyroradius is

$$r_{\rm g} \sim 0.02 \text{ AU} = 10^{-6} \text{ pc} \tag{13}$$

so $r_{\rm g} \ll$ solar system, interstellar scales
cosmic rays definitely do not move in straight lines

possible exception: $r_{\rm g} \gtrsim R_{\rm MW} \sim 10$ kpc
for $p \gtrsim 10^{10}$ GeV $= 10^{19}$ eV
$\rightarrow$ "ultra-high-energy cosmic rays"
`www: arrival directions for UHECR`

returning to typical cosmic rays: gyrofrequency

$$\nu_{\rm g} = \frac{\omega_{\rm g}}{2\pi} = \frac{eB}{2\pi\gamma mc} = 2.8\text{Hz}\ \gamma^{-1} \left(\frac{B}{1\ \mu\text{Gauss}}\right) \left(\frac{m_e}{m}\right) \tag{14}$$

Q: implications for electrons? protons?

gyrofrequency for mildly relativistic electrons:
cyclotron frequency $\nu_{\rm g} \sim few$ Hz
$\rightarrow$ very slow! huge wavelengths
if radiation is only at this frequency
would seem undetectable

but we will see: for relativistic electrons
radiation is at much higher frequencies!
synchrotron radiation

even so, low gyrofrequency hints that *radio* frequencies
likely to be important for synchrotron emission
`www: Kepler supernova remnant at 6 cm (VLA)`
Q: implications of intensity pattern?

Q: how to evaluate emitted synchrotron power from CR electrons?

Synchrotron Power

Lorentz-invariant power emitted from accelerated charge is

$$P = \frac{2}{3}\frac{q^2}{c^3} a' \cdot a' = \frac{2}{3}\frac{q^2}{c^3}(a'^2_{\parallel} + a'^2_{\perp}) \quad (15)$$

$$= \frac{2}{3}\frac{q^2}{c^3}\, \gamma^4\, (a^2_{\perp} + \gamma^2 a^2_{\parallel}) \quad (16)$$

for our case of circular motion: $a_{\parallel} = 0$,and
$a_{\perp} = \omega_B v_{\perp}$, so

$$P = \frac{2}{3}\frac{q^2}{c^3}\, \gamma^4\, \frac{q^2 B^2}{\gamma^2 m^2 c^2} = \frac{2}{3} r_0^2\; c\; \gamma^2 \beta_{\perp}^2 B^2 \quad (17)$$

but electron distribution is isotropic
so must *average over* distribution of *pitch angle* $\hat{v} \cdot \hat{B} = \cos\alpha$

$$\left\langle \beta_{\perp}^2 \right\rangle = \frac{\beta^2}{4\pi} \int \sin^2\alpha \; d\Omega = \frac{2}{3}\beta^2 \quad (18)$$

total synchrotron power from isotropic electrons

$$P = \left(\frac{2}{3}\right)^2 r_0^2 \; c \; \gamma^2 \beta B^2 = \frac{4}{3} \sigma_{\mathsf{T}} \; c \; \beta^2 \gamma^2 \; u_B \tag{19}$$

where $\sigma_{\mathsf{T}} = 8\pi r_0^2/3$ and $u_B = B^2/8\pi$

Q: how to find the spectrum of synchrotron radiation?

Q: why is it non-trivial? hint–think of relativistic circular motion