Astro 501: Radiative Processes
Lecture 20
March 4, 2013

Announcements:

- **Problem Set 6** available, due Friday 5pm
- Midterm Exam: grading elves hard at work
- free advice: you'll be glad if you don't miss class Wednesday

Before exam: began synchrotron

Q: what is cyclotron radiation? synchrotron radiation?

Q: characteristic scales?

synchrotron radiation: relativistic charged particle in uniform magnetic field

moves in circle with **gyroradius**

$$r_{\mathrm{g}} = \frac{mc\gamma v_\perp}{qB} = \frac{cp_\perp}{qB} \simeq 10^{-6}\ \mathrm{pc}\ \left(\frac{cp_\perp}{1\ \mathrm{GeV}}\right)\ \left(\frac{1\ \mu\mathrm{Gauss}}{B}\right) \tag{1}$$

non-relativistic gyrofrequency

$$\nu_{\mathrm{g}} = \frac{\omega_{\mathrm{g}}}{2\pi} = \frac{eB}{2\pi\gamma mc} = 2.8\mathrm{Hz}\ \gamma^{-1}\ \left(\frac{B}{1\ \mu\mathrm{Gauss}}\right)\ \left(\frac{m_e}{m}\right) \tag{2}$$

gyrofrequency for mildly relativistic electrons:
cyclotron frequency $\nu_{\mathrm{g}} \sim few$ Hz

full relativistic frequency

$$\omega_B = \frac{\omega_{\mathrm{g}}}{\gamma} = \frac{qB}{\gamma mc} \tag{3}$$

Synchrotron Power

Lorentz-invariant power emitted from accelerated charge is

$$P = \frac{2}{3}\frac{q^2}{c^3}\,\gamma^4\,(a_\perp^2 + \gamma^2 a_\parallel^2) \tag{4}$$

for our case of circular motion: $a_\parallel = 0$,and

$a_\perp = \omega_B v_\perp$, so

$$P = \frac{2}{3}\frac{q^2}{c^3}\,\gamma^4\,\frac{q^2 B^2}{\gamma^2 m^2 c^2} = \frac{2}{3} r_0^2\; c\; \gamma^2 \beta_\perp^2 B^2 \tag{5}$$

but electron distribution is isotropic

so must *average over* distribution of *pitch angle* $\hat{v} \cdot \hat{B} = \cos\alpha$

$$\left\langle \beta_\perp^2 \right\rangle = \frac{\beta^2}{4\pi} \int \sin^2\alpha \; d\Omega = \frac{2}{3}\beta^2 \tag{6}$$

total synchrotron power from isotropic electrons

$$P = \left(\frac{2}{3}\right)^2 r_0^2 \; c \; \gamma^2 \beta B^2 = \frac{4}{3} \sigma_\mathrm{T} \; c \; \beta^2 \gamma^2 \; u_B \tag{7}$$

where $\sigma_\mathrm{T} = 8\pi r_0^2/3$ and $u_B = B^2/8\pi$

another awesome astrophysical example: radio galaxies

Q: what's that?

`www: radio images of Cygnus A, Centaurus A`

Q: how to find the spectrum of synchrotron radiation?

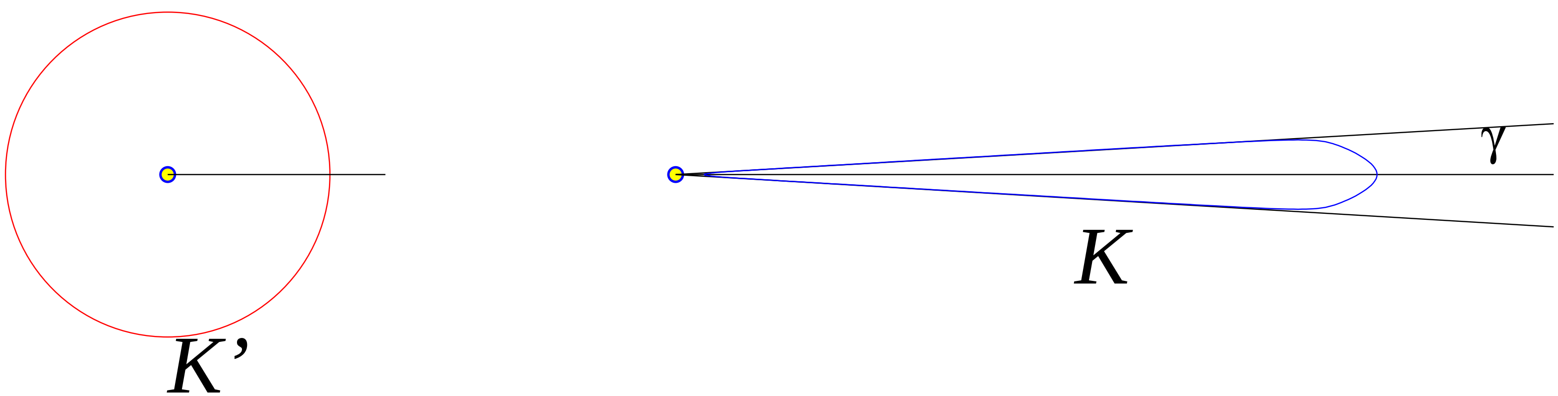
Q: why is it non-trivial? hint–think of relativistic circular motion

Spectrum of Synchrotron Radiation: Order of Magnitude

key issue:

radiation from a relativistic accelerated particle is *beamed*

into forward cone of opening angle $\theta_{\text{beam}} \sim 1/\gamma$

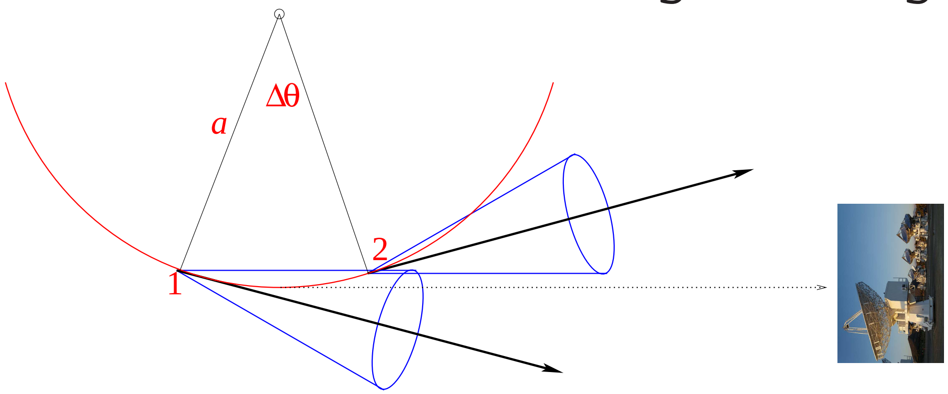


so observer receives pulses or "flashes" of radiation

spread over narrow timescale $\ll 2\pi/\omega_B$

sharply peaked signal in time domain

$\Rightarrow$ *broad signal in frequency domain* consider relativistic charge moving in circle of radius a



observer only sees emission over angular range

$$\Delta\theta \simeq 2\theta_{\text{beam}} \simeq \frac{2}{\gamma} \tag{8}$$

representing a path length

$$\Delta s = a \ \Delta\theta = \frac{2a}{\gamma} \tag{9}$$

gyroradius is $a = v/\omega_B \sin\alpha$, so

$$\Delta s \simeq \frac{2v}{\gamma\omega_B \sin\alpha} \tag{10}$$

if the particle *passes point 1* at t_1 and point 2 at t_2

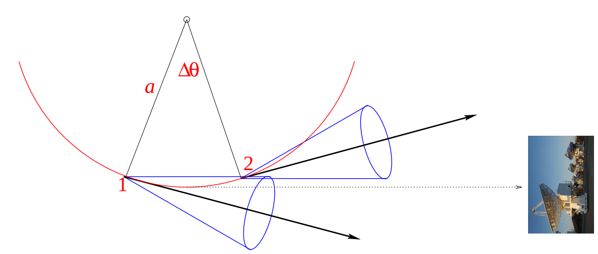
$\Delta s = v(t_2 - t_1)$, and

$$\Delta t = t_2 - t_1 \simeq \frac{2}{\gamma\omega_B \sin\alpha} \tag{11}$$

what is *arrival time* of radiation?

note that point 2 is closer than point 1 by $\approx \Delta s$

$$\begin{aligned}
\Delta t^{\mathrm{arr}} = t_2^{\mathrm{arr}} - t_1^{\mathrm{arr}} &= \Delta t - \frac{\Delta s}{c} \\
&= \Delta t \left(1 - \frac{v}{c}\right) \\
&= \frac{2}{\gamma\omega_B \sin\alpha} \left(1 - \frac{v}{c}\right)
\end{aligned}$$



radiation arrive time duration

$$\Delta t^{\rm arr} = \frac{2}{\gamma \omega_B \sin\alpha}\left(1 - \frac{v}{c}\right) \tag{12}$$

but note that $1 - v/c \approx 1/2\gamma^2$ for relativistic motion *Q:why?*

and thus radiation arrives in pulse of duration

$$\Delta t^{\rm arr} \approx \frac{1}{\gamma^3 \omega_B \sin\alpha} \tag{13}$$

shorter than ω_B^{-1} by factor γ^3!

define **critical frequency**

$$\omega_{\rm c} \equiv \frac{3}{2}\gamma^3 \omega_B \sin\alpha = \frac{3}{2}\gamma^2 \, \frac{qB\sin\alpha}{mc} = \frac{3}{2}\gamma^2 \, \omega_{\rm g} \sin\alpha \tag{14}$$

∞

$$\nu_{\rm c} = \frac{\omega_{\rm c}}{2\pi} = \frac{3}{4\pi}\gamma^3 \omega_B \sin\alpha \tag{15}$$

Q: will radiation spectrum cut off above or below $\omega_{\rm c}$*?*

critical frequency

$$\nu_{\text{c}} = \frac{3}{4\pi}\gamma^3 \omega_B \sin\alpha \sim \frac{1}{\Delta t^{\text{arr}}} \tag{16}$$

Fourier transform of pulse Δt^{arr} broad up to ν_{c}
and should cut off above this

numerically:

$$\nu_{\text{c}} = 25 \text{ MHz} \left(\frac{E_e}{1 \text{ GeV}}\right)^2 \left(\frac{B}{1 \ \mu\text{Gauss}}\right) \sin\alpha \tag{17}$$

Q: lessons? irony?

critical = characteristic frequency $\nu_c \sim 25$ MHz $(E_e/1\ \text{GeV})^2$
typical cosmic-ray electrons emit in the observable *radio*
$\rightarrow$ *high-energy* electrons can emit *low-frequency* radiation!

expect synchrotron power of form $P(\omega) \sim P/\omega_{\text{c}}\ F(\omega/\omega_{\text{c}})$
with dimensionless function $F(x)$

- should be peaked at $x \sim 1$, then drop sharply
- can only be gotten from an honest calculation!

note: $P \propto \gamma^2$ but $\omega_{\text{c}} \propto \gamma^2 \rightarrow P/\omega_{\text{c}}$ indep of γ

for a particle with a fixed v and γ,
conventional to define synchrotron spectrum as

$$\frac{dP}{d\omega} = P(\omega) = \frac{\sqrt{3}}{2\pi}\frac{q^3 B \sin\alpha}{mc^2} \, F\left(\frac{\omega}{\omega_{\rm c}}\right) \tag{18}$$

with $\omega_{\rm c} \propto \gamma^2$

where the *synchrotron function* (derived in RL) is

$$F(x) = x\int_x^\infty K_{5/3}(t)\ dt \longrightarrow \begin{cases} \frac{4\pi}{\sqrt{3}\Gamma(1/3)}\left(\frac{x}{2}\right)^{1/3} & x \ll 1 \\ \left(\frac{\pi}{2}\right)^{1/2} e^{-x} x^{1/2} & x \gg 1 \end{cases} \tag{19}$$

with $K_{5/3}(x)$ the modified Bessel function of order 5/3
$\rightarrow$ *sharply peaked* at $\omega_{\rm max} = x_{\rm max}\omega_{\rm c} = 0.29\omega_{\rm c}$
`www: plot of synchrotron function`

Q: so is this the spectrum we would see for real CR es?

for a **single** electron γ
emission spectrum is synchrotron function $F(\omega/\omega_c)$
sharply peaked near $\omega_c \propto \omega_g \gamma^2$

but the *population* of cosmic-ray electrons
has a *spectrum* of energies and thus of γ

resulting synchrotron spectrum is

- *superposition* of peaks $\propto \gamma^2$,
- *weighted by electron energy spectrum*

Q: what if CRs had two energies? N energies?

Q: what does the real spectrum look like?

Q: what's the synchrotron spectral shape for the ensemble of all electron energies?

recall: cosmic-ray electron spectrum well-fit by *power law*
so number of particles with energy in $(E, E + dE)$ is

$$N(E)\ dE = C\ E^{-p}\ dE \tag{20}$$

and so

$$N(\gamma)\ d\gamma = C'\ \gamma^{-p}\ d\gamma \tag{21}$$

note that for a single electron v and γ
$P(\omega) \propto F(\omega/\omega_{\rm c})$ and $\omega_{\rm c} = \omega_{\rm g}\gamma^2$

so integrating over full CR spectrum means

$$\begin{aligned} \langle P(\omega) \rangle &= \int P(\omega)\ N(\gamma)\ d\gamma && (22) \\ &= C' \int P(\omega)\ \gamma^{-p}\ d\gamma && (23) \\ &\propto \int F\left(\frac{\omega}{\omega_{\rm g}\gamma^2}\right)\ \gamma^{-p}\ d\gamma && (24) \end{aligned}$$

Q: strategy?

$$\langle P(\omega)\rangle \propto \int F\left(\frac{\omega}{\omega_{\rm g}\gamma^2}\right)\ \gamma^{-p}\ d\gamma \quad (25)$$

change integration variable to $x = \omega/\omega_{\rm c} = \gamma^{-2}\omega/\omega_{\rm g}$
$\rightarrow \gamma = (\omega x/\omega_{\rm g})^{-1/2}$, and $d\gamma = -(\omega/\omega_{\rm g})^{-1/2} x^{-3/2} dx$

$$\langle P(\omega)\rangle \propto \left(\frac{\omega}{\omega_{\rm g}}\right)^{-(p-1)/2} \int F(x)\ x^{(p-3)/2}\ dx \quad (26)$$

and so

$$\langle P(\omega)\rangle \propto \omega^{-(p-1)/2} = \omega^{-s} \quad (27)$$

with **spectral index** $s = (p-1)/2$

even though each electron energy $\rightarrow$ peaked emission
average over power-law electron distribution
$\rightarrow$ power-law synchrotron emission

full expression for power-law electron spectrum of the form $dN/d\gamma = C\gamma^{-p}$

$$4\pi j_{\rm tot}(\omega) = \frac{\sqrt{3}q^3 CB\sin\alpha}{2(p+1)\pi mc^2}\,\Gamma\left(\frac{p}{4}+\frac{9}{12}\right)\,\Gamma\left(\frac{p}{4}-\frac{1}{12}\right)\,\left(\frac{mc\omega}{3qB\sin\alpha}\right)^{-(p-1)/2} \tag{28}$$

with $\Gamma(x)$ the gamma function, with $\Gamma(x+1) = x\,\Gamma(x)$

Q: overall dependence on B? does this make sense?

Q: expected spectral index?

Q: do you expect the signal to be polarized? how?