Astro 501: Radiative Processes
Lecture 21
March 6, 2013

Announcements:

- **Problem Set 6** due Friday 5pm
- Midterm Exam: grading elves hard at work

Last time: synchrotron spectrum

Q: spectrum for isotropic, monoenergetic electrons?

Q: spectrum for electrons with a power-law energy distribution?

define **critical frequency**

$$\omega_{\rm c} \;\equiv\; \frac{3}{2}\gamma^3 \omega_B \sin\alpha = \frac{3}{2}\gamma^2 \, \frac{qB\sin\alpha}{mc} = \frac{3}{2}\gamma^2 \, \omega_{\rm g} \sin\alpha \qquad (1)$$

$$\nu_{\rm c} \;=\; \frac{\omega_{\rm c}}{2\pi} = \frac{3}{4\pi}\gamma^3 \omega_B \sin\alpha \qquad (2)$$

emission spectrum is synchrotron function $F(\omega/\omega_{\rm c})$

sharply peaked near $\omega_{\rm c} \propto \omega_{\rm g}\gamma^2$

full expression for power-law electron spectrum of the form $dN/d\gamma = C\gamma^{-p}$

$$4\pi j_{\mathrm{tot}}(\omega) = \frac{\sqrt{3}q^3 CB\sin\alpha}{2(p+1)\pi mc^2}\,\Gamma\left(\frac{p}{4}+\frac{9}{12}\right)\,\Gamma\left(\frac{p}{4}-\frac{1}{12}\right)\,\left(\frac{mc\omega}{3qB\sin\alpha}\right)^{-(p-1)/2} \tag{3}$$

with $\Gamma(x)$ the gamma function, with $\Gamma(x+1) = x\,\Gamma(x)$

Q: expected spectral index?

Q: do you expect the signal to be polarized? how?

Polarization of Synchrotron Radiation

for an electron with a single pitch angle $\tan\alpha = v_\perp / v_\parallel$
$\rightarrow$ circular motion around field line
$\rightarrow$ radiation circularly polarized orthogonal to $\vec{B}$
and elliptically polarized at arbitrary angles

but with distribution of pitch angles α,
elliptical portion cancels out $\rightarrow$ partial **linear polarization**

polarization strength varies with projected angle
of magnetic field on sky
more power orthogonal to projected field direction
$\rightarrow$ net linear polarization, detailed formulae in RL

averaging over power-law distribution of electron energies
partial polarization is $\Pi = (p+1)/(p+7/3)$
and so $\Pi = 3/4$ for $p = 3$: highly polarized!

Transition from Cyclotron to Synchrotron

How and why are the emission spectra so different
for cyclotron (non-relativistic) vs synchrotron (relativistic)?

recall: in either case, electron motion is *strictly periodic*
with angular frequency

$$\omega_B = \frac{qB \sin \alpha}{mc\gamma} \tag{4}$$

Q: nature of Fourier spectrum of received field?

Q: Fourier spectrum of emission for single pitch angle?

Q: spectrum in nonrelativistic case $\gamma \to 1$?

Q: spectrum in mildly relativistic case?

electron motion at fixed α strictly periodic with ω_B
$\rightarrow$ received field also strictly periodic
$\rightarrow$ Fourier transform of field is nonzero only for
discrete *series* of frequencies $m\omega_B$, $m \in 1, 2, \ldots$

and thus received radiation also is a Fourier series in ω_B

cyclotron = nonrelativistic case: see field $E = E_0 \cos \omega_B t$
Fourier series has *one term*: the fundamental frequency ω_B

when mildly relativistic: Doppler effects add harmonic at $2\omega_B$
and electric field shape modified to sharper, narrower peak

going to strongly relativistic: many harmonics excited
series "envelope" approaches $F(\omega/\omega_c)$
electric field $\rightarrow$ very sharp, very narrow peak

with distribution of pitch angles:
"spaces" in series filled in $\rightarrow$ continuous spectrum

Synchrotron Self-Absorption

Recall strategy so far:

- calculate emission coefficient j_ν
- remember Kirchoff's law $j_\nu = \alpha_\nu \; B_\nu(T)$
- solve for $\alpha_\nu = j_\nu / B_\nu(T)$

We have already found

Q: why won't this work here?

Q: what do we need to do? hint–how did we handle a two-level system?

Kirchoff's law is only good for a *thermal* system
where emitter and absorber particles are nonrelativistic
and have Maxwell-Boltzmann energy/momentum distribution

here: electrons are relativistic and nonthermal

really: Kirchoff is example of *detailed balance*
$\rightarrow$ in equilibrium, emission and absorption rates are the same $\rightarrow$
this still applies in nonthermal case

recall from *2-level* system, with $E_2 = E_1 + h\nu$

$$\alpha_\nu \stackrel{\text{2-level}}{=} \frac{h\nu}{4\pi} \left[n(E_1) B_{12} - n(E_2) B_{21} \right] \; \phi(\nu) \tag{5}$$

Q: physical interpretation of $n(E_1)$? B_{12}? B_{21}? $\phi(\nu)$?

Q: how should this be modified for synchrotron electrons?

in *2-level system*, emission at frequency ν
arises from *unique energy level spacing* $E_2 = E_1 + h\nu$

but cosmic ray electrons have *continuous energy spectrum*
$\rightarrow$ emission at ν can arise from *any two energies*:
generalized to

$$\alpha_\nu = \frac{h\nu}{4\pi} \sum_{E_1} \sum_{E_2} \left[n(E_1) B_{12} - n(E_2) B_{21} \right] \ \phi_{21}(\nu) \tag{6}$$

- with $\phi_{21}(\nu) \rightarrow \delta[\nu - (E_2 - E_1)/h]$
- first term: true absorption
- second term: stimulated emission

the goal: recast this in terms of what we know
synchrotron emission j_ν

we have

$$\alpha_\nu = \frac{h\nu}{4\pi} \sum_{E_1} \sum_{E_2} [n(E_1) B_{12} - n(E_2) B_{21}] \;\; \phi_{21}(\nu) \tag{7}$$

use Einstein relations, good for thermal and nonthermal

- spontaneous emission rate from state E_2: $A_{21} = 2h\nu^3 B_{21}/c^2$
- absorption and stimulated emission: $B_{21} = B_{12}$

note that spontaneous *emission* is what we know!
we have found synchrotron power $P(\nu, E_2) = 2\pi P(\omega)$,
with E_2 the radiating electron's energy

$$P(\nu, E_2) = h\nu \sum_{E_2} A_{21} \;\; \phi_{21}(\nu) \tag{8}$$

now impose Einstein conditions and simplify

Q: role of ϕ_{21} and double sum $\sum_{E_1} \sum_{E_2}$?

profile function $\phi_{21}(\nu) \to \delta(E_2 - E_1 - h\nu)$
fixes E_1 for a given E_2 and ν
and double sum $\to$ single sum

$$\alpha_\nu = \frac{c^2}{8\pi h\nu^3} \sum_{E_2} [n(E_2 - h\nu) - n(E_2)] \; P(\nu, E_2) \tag{9}$$

so far: schematic sum over electron energies
but really a continuum

recall: in each phase space cell h^3
• number of electron states with momentum p is $g_e \; f(p)$
• volume density of states in momentum space volume is d^3p/h^3
and thus

$$\alpha_\nu = g_e \frac{c^2}{8\pi h\nu^3} \frac{1}{h^3} \int [f(p_2^*) - f(p_2)] \; P(\nu, E_2) \; d^3p_2 \tag{10}$$

where p_2^* is the momentum corresponding to energy $E_2 - h\nu$

Q: how is f related to electron spectum $N(E)$?

number of electrons per unit volume
with energy in $(E, E + dE)$ is $N(E)\ dE$

but this means that

$$N(E)\ dE = \frac{4\pi\ g_e}{h^3}\ p^2\ f(p)\ dp \tag{11}$$

and for ultrarelativistic electrons, $E = cp$

thus we have

$$\alpha_\nu = \frac{c^2}{8\pi h\nu^3} \int \left[\frac{N(E - h\nu)}{(E - h\nu)^2} - \frac{N(E)}{E^2} \right]\ E^2\ P(\nu, E)\ dE \tag{12}$$

and since $h\nu \ll E$, expand to first order

$$\alpha_\nu = -\frac{c^2}{8\pi\nu^2} \int dE\ P(\nu, E)\ E^2\ \partial_E \left[\frac{N(E)}{E^2} \right] \tag{13}$$

and for a power-law $N(E) \propto E^{-p}$, we have

$$-\ E^2 \partial_E \left[\frac{N(E)}{E^2} \right] = (p + 2) \frac{N(E)}{E} \tag{14}$$

Synchrotron Absorption

finally then

$$\alpha_\nu = (p+2)\frac{c^2}{8\pi\nu^2}\int dE\ P(\nu,E)\ \frac{N(E)}{E} \tag{15}$$

note frequency dependence:

- prefactor ν^{-2}
- integral $\int dE\ P(\nu)N(E)/E \sim dE\ P(\nu)E^{-(p+1)} \sim \nu^{-p/2}$

net scaling: $\alpha_\nu \propto \nu^{-(p+4)/2}$

full result

$$\begin{aligned}\alpha_\nu \;\; = \;\; & \frac{\sqrt{3}}{8\pi}\Gamma\left(\frac{3p+2}{12}\right)\Gamma\left(\frac{3p+22}{12}\right) \\ & \left(\frac{3q}{2\pi m^3c^5}\right)^{p/2}\left(\frac{q^3C}{m}\right)\ (B\sin\alpha)^{(p+2)/2}\ \nu^{-(p+4)/2}\end{aligned}$$

Source Function

source function

$$S_\nu = \frac{j_\nu}{\alpha_\nu} \propto \frac{\nu^{-(p-1)/2}}{\nu^{-(p+4)/2}} = \nu^{5/2} \tag{16}$$

to see this, recall that

$$j_\nu \ \sim \ \int dE \ N(E) \ P(\nu) \tag{17}$$

$$\alpha_\nu \ \sim \ \nu^{-2} \int dE \ \frac{N(E)}{E} \ P(\nu) \tag{18}$$

thus source function has

$$S_\nu \sim \nu^2 \bar{E} \tag{19}$$

with typical electron energy $\bar{E} = m\bar{\gamma}$ for freq ν
but $\nu(E) \approx \nu_{\rm c}(E) \sim E^2$, so $\bar{E} \propto \nu^{1/2}$
and thus $S_\nu \sim \nu^{5/2}$ *independent of electron spectral index*

Synchrotron Radiation: the Big Picture

for relativistic electrons with power-law energy distribution

emission coefficient

$$j_\nu \propto \nu^{-(p-1)/2} \tag{20}$$

absorption coefficient

$$\alpha_\nu \propto \nu^{-(p+4)/2} \tag{21}$$

source function (note nonthermal character!)

$$S_\nu \propto \nu^{5/2} \tag{22}$$

Q: optical depth vs ν? implications?
Q: spectrum of a synchrotron emitter?

`www: awesome example: pulsar wind nebulae`
young pulsars are spinning down
much of rotational energy goes into relativistic wind
which collides with the supernova ejecta an emits synchrotron