Astro 501: Radiative Processes
Lecture 32
April 10, 2013

Announcements:
- **Problem Set 10** due 5pm Friday April 19
- ICES to be available online – please!

Last time: we reached the summit!
quantum mechanical expressions for
- Einstein $A$ and $B$, or equivalently
- absorption cross section

*Q: key quantum ingredient?*

Today: the payoff
the astrophysics of absorption lines; draws on
Draine, *Physics of the Interstellar and Intergalactic Medium*
Padmanabhan, *Theoretical Astrophysics*

the Einstein coefficients in the electric dipole approximation are:

- true *absorption*

$$B_{\ell u} = \frac{8\pi^2}{3c\hbar^2} |d_{\ell u}|^2 = \frac{32\pi^4}{3ch^2} |d_{\ell u}|^2 \tag{1}$$

for *non-degenerate atomic levels* with $g_\ell = g_u = 1$ we have

- *stimulated emission*

$$B_{u\ell} = B_{\ell u} \tag{2}$$

- *spontaneous emission*

$$A_{u\ell} = \frac{4\omega_{u\ell}^3 \ |d_{u\ell}|^2}{3c^3\hbar} = \frac{64\pi^4 \ \nu_{u\ell}^3 \ |d_{u\ell}|^2}{3c^3 h} \tag{3}$$

notice that

$$\hbar\omega_{u\ell} \ A_{u\ell} = \frac{4\omega_{u\ell}^4 \ |d_{u\ell}|^2}{3c^3} \tag{4}$$

*Q: what does this represent physically? why is this awesome?*

Einstein $A_{u\ell}$ is $u \to$ transition probability per unit time
$\to$ spontaneous emission rate per atom in state $u$

thus $E_{u\ell}\ A_{u\ell} = \hbar\omega_{u\ell}\ A_{u\ell}$ is energy emission rate
$=$ power emitted per atom

$$P_{u\ell} = \frac{4\omega_{u\ell}^4\ |d_{u\ell}|^2}{3c^3} \tag{5}$$

but recall our classical Larmor dipole result

$$P = \frac{4\ |\ddot{\vec{d}}|^2}{3c^2} = \frac{4\ \omega^4\ |\vec{d}|^2}{3c^2} \tag{6}$$

*quantum result has same form*, with $|\vec{d}| = |d_{u/\ell}| = |\langle \ell | e\vec{r} | u \rangle|$

results often expressed in terms of oscillator strength $f$
*Q: why? what's $f$? typical values?*

If the electron moves as a *damped classical oscillator*
with natural (resonant) frequency $\omega_0$
then (PS10) absorption rate is $B_{\ell u}^{\text{classcial}}\ J(\nu_{\ell u})$ with

$$B_{\ell u}^{\text{classical}} = \frac{4\pi^2 e^2}{h\nu_{\ell u}\ m_e c} \tag{7}$$

it is thus convenient write

$$B_{\ell u} \equiv f_{\ell u}\ B_{\ell u}^{\text{classical}} \tag{8}$$

$$\sigma_{\ell u}(\nu) = \frac{\pi e^2}{m_e c}\ f_{\ell u}\ \phi(\nu) \tag{9}$$

sum rule: $\sum_{j\ \text{final}} f_{ij} = N$
for strong allowed transitions, $f \sim 1$

# Shape of Spectral Lines

consider a transition $u \to \ell$
*Q: most naïve guess for line profile* $\phi(\nu)$

real astrophysical spectra show wide range profiles
with nonzero observed widths

`www: solar spectrum` try $(\lambda_i, \Delta\lambda) = (6500, 100)$nm; $(4043, 5)$; $(6704, 8)$

`www: spectrum of mystery star`
*Q: how is this star different from the Sun?*
*hint–look at the continuum*

`www: spectrum of interstellar matter`
*Q: how is this gotten? how do we know the lines are ISM?*



*Q: reason(s) for nonzero observed linewidths?*

# Linewidths

naïvely: in transition $u \to \ell$, *energy conservation* requires
$h\nu = E_u - E_\ell \equiv h_{u\ell}$, so $\phi_{\text{naive}}(\nu) = \delta(\nu - \nu_{u\ell})$: *zero width!*

But real observed linewidths are nonzero, for several reasons

- *intrinsic width*
quantum effect, due to nonzero transition probability

- *thermal broadening*
thermal motion of absorbers $\to$ Doppler shifts

- *collisional broadening*
absorber collisions add to transition probability

- *instrumental resolution*
real spectrographs have finite resolving power
$R = \lambda/\Delta\lambda = \nu/\Delta\nu \overset{\text{Keck}}{\sim} 30,000$

# Intrinsic Linewidth

in real atoms, any excited state $u$ has nonzero transition rate to lower levels: $\Gamma_u = 1/\tau_u$, with $\tau_u$ the state *lifetime*

thus: state $u$ is only populated for timescales $\delta t \sim \tau_u$

but in quantum mechanics, over finite time $\Delta t$,
*energy* only determined to within finite resolution

$$\Delta E \ \Delta t \gtrsim \frac{\hbar}{2} \tag{10}$$

the **energy-time uncertainty relation**

thus state $u$, level energy $E_u$ has intrinsic spread
$\delta E_u \sim \hbar/\tau_u = \hbar\Gamma_u$
*Q: consequence for line profile?*

level $u$ energy intrinsic spread $\delta E_u \sim \hbar/\tau_u = \hbar\Gamma_u$
so for $u \to \ell$, transition frequency $\nu_{u\ell} = (E_u - E_\ell)/h$
has natural or *intrinsic width* $\delta\nu_{n\ell} = \Gamma_{u\ell} = \Gamma_u + \Gamma_\ell$

level lifetimes related to Einstein $A$ = decay rates:

$$\Gamma_u = \Gamma_{u\to\text{anything}} = \sum_{u\to\text{allowed } j} A_{uj} \tag{11}$$

where sum is over *all energetically allowed* transitions from $u$

for *damped classical oscillator*, damping $\Gamma\dot{x}$
leads (PS10) to absorption cross section

$$\sigma_{\ell u}(\nu) = \frac{2\pi e^2}{m_e c} \frac{\Gamma/2}{(\omega-\omega_0)^2 + (\Gamma/2)^2} = \frac{\pi e^2}{m_e c} \frac{4\Gamma}{16\pi^2(\nu-\nu_0)^2 + \Gamma^2}$$



*Q: behavior at* $\nu = \nu_0$*?* $\nu \gg \nu_0$*? what about a real atomic transition* $u \to \ell$*?*

for a damped classical oscillator, we have

$$\sigma(\nu) = \pi e^2/m_e c \ \phi(\nu) = B_{\text{classical}} \ \phi(\nu) \tag{12}$$

with profile function (normalized to $\int \phi(\nu) \ d\nu = 1$) of

$$\phi(\nu) = \frac{4\Gamma}{16\pi^2(\nu - \nu_0)^2 + \Gamma^2}$$

a real atomic transition $u \to \ell$ has same properties but with overall factor of oscillator strength:

$$\sigma_{u\ell}(\nu) = \pi e^2/m_e c \ f_{u\ell} \ \phi_{u\ell}(\nu) = B_{\text{classical}} \ f_{u\ell} \ \phi(\nu) \tag{13}$$

with *Lorentzian* profile shape

$$\phi_{u\ell}^{\text{intrinsic}}(\nu) = \frac{4\Gamma_{u\ell}}{16\pi^2(\nu - \nu_{u\ell})^2 + \Gamma_{u\ell}^2}$$

full width at half-maximum: $(\Delta\nu)_{\text{FWHM}} = \Gamma_{u\ell}/2\pi$

note that line profiles and linewidths are often expressed in line-of-sight *velocity* units

motivated by the non-relativistic Doppler formula, we have

$$v(\nu) = \frac{\nu - \nu_{u\ell}}{\nu_{u\ell}}\ c \tag{14}$$

so that $v(\nu_{u\ell}) = 0$ at line center

thus the FWHM in velocity units is

$$(\Delta v)_{\rm FWHM} = \frac{(\Delta \nu)_{\rm FWHM}}{\nu_{u\ell}} c = \frac{\Gamma_{u\ell}\lambda_{u\ell}}{2\pi} \tag{15}$$

for optical and UV transitions, intrinsic linewidths generally small:
for Lyman-$\alpha$, $(\Delta v)_{\rm FWHM,Ly\alpha} = 0.0121$ km/s
*Q: implications?*

# Thermal Linewidth

intrinsic linewidths are generally narrow
so other broadening effects can be important

*thermal motion* of atoms leads to Doppler shifts
of incident spectra as seen by the atoms
so absorption occurs "off resonance"

a *Gaussian distribution* of line-of-sight velocities
has velocity probability distribution

$$p(v)\ dv = \frac{1}{\sqrt{2\pi}\sigma_v} e^{-(v-v_0)^2/2{\sigma_v}^2} \equiv \frac{1}{\sqrt{\pi}b} e^{-(v-v_0)^2/b^2}\ dv \qquad (16)$$

where $v_0$ is the bulk or "systemic" velocity along sightline
$\sigma_v = b/\sqrt{2}$ is the *velocity dispersion*



*Q:* $v_0$, $\sigma_v$, *and* $b$ *for thermal gas at rest??*

a thermal gas at $T$ of particles with mass $m$, and at rest in bulk, has

$$p_T(v)\ dv = \sqrt{\frac{m}{2\pi kT}} e^{-mv^2/2kT} \tag{17}$$

from which we identify

$$v_0 = 0 \tag{18}$$

$$\sigma_v = v_T \equiv \sqrt{\frac{kT}{m}} = 9.12 \text{ km/s} \left(\frac{T}{10^4 \text{ K}}\right) \left(\frac{1 \text{ amu}}{m}\right) \tag{19}$$

$$b = \sqrt{\frac{2kT}{m}} \tag{20}$$

*Q: implications of numerical result?*

*Q: how to combine intrinsic and thermal broadening?*

# Voigt Profile

in general both intrinsic and thermal broadening present
and so resulting line profile includes both effects

observed profile is *weighted average*
of natural/intrinsic width with Doppler shifted center

$$\nu'_{u\ell} = \left(1 - \frac{v}{c}\right) \nu_{u\ell} \quad (21)$$

giving the **Voigt profile**

$$\phi_{\rm Voigt}(\nu) = \frac{1}{\sqrt{\pi}\ b} \int e^{-v^2/b^2} \ \frac{4\Gamma_{u\ell}}{16\pi^2 \left[\nu - (1 - v/c)\nu_{u\ell}\right]^2 + \Gamma_{u\ell}^2} \ dv$$

integral has no simple analytic result

 *Q: simple and interesting approximation?*

we saw that for astrophysical situations, often intrinsic linewidths $(\Delta v)_{\mathrm{FWHM}} \ll b$ thermal linewidths

simple approximation: intrinsic absorption is $\delta$-function $\phi^{\mathrm{intrinsic}}(\nu) \to \delta[\nu - (1 - v/c)\nu_{u\ell}]$

this gives a thermally-dominated Voigt profile

$$\phi_{\mathrm{Voigt}}(\nu) \;\to\; \phi_T(\nu) = \frac{1}{\sqrt{\pi}} \frac{c}{\nu_{u\ell}\, b} \;\exp\left[-\frac{v(\nu)^2}{b^2}\right] \tag{22}$$

valid in the *"thermal core"* $\nu - \nu_{u\ell} \ll \Gamma_{u\ell}$, with

$$v(\nu) \equiv \left(\frac{\nu - \nu_{u\ell}}{\nu_{u\ell}}\right) c \tag{23}$$

for $\nu - \nu_{u\ell} \gg b$, in the *"damping wings,"* we have

$$\phi_{\mathrm{Voigt}}(\nu) \approx \frac{1}{4\pi^2} \frac{\Gamma_{u\ell}}{(\nu - \nu_{u\ell})^2} \tag{24}$$

*Q: sketch of $\phi_{\mathrm{Voigt}}(\nu)$? of $\sigma_{u\ell}(\nu)$?*

so imagine we can resolve a strong absorption line
and measure the shape vs $\nu$ or $\lambda$ to high precision

*Q: what will we see?*

*Q: what will we learn?*

*Q: what if the line is not very strong?*

*Q: what if we only have moderate spectral resolution?*

`www: overview of the optical solar spectrum`
*Q: what are we seeing?*

# Collisional Linewidth

if particle densities are high, atomic collisions are rapid
and can drive transitions $u \leftrightarrow \ell$

thus there is a nonzero collision rate $\Gamma_{\rm coll}$ per atom
where $\Gamma_{\rm coll} = n\ \sigma_{\rm coll} v$

heuristically: this decreases excited state lifetimes
and thus adds to energy uncertainty

so total transition rate is $\Gamma_{\rm int} + \Gamma_{\rm coll}$: collisions add damping,
in density- and temperature-dependent way

thus collisional broadening a measure of density and temperature
thus also know as "pressure broadening"