Astro 501: Radiative Processes
Lecture 4
Jan 21, 2013

Announcements:

- **Problem Set 1** at **start of class** next Friday
  Erratum: err, we're ASTR 501 in 2013, not Cosmo in 2010!
  Erratum: find the source function S_ν for 2(c), not 2(a)
  Erratum: Problem 3c: cite full of typos, should read
  Hoffmann, Tappert, et al (1998) `1998A&AS..128..417H`
  and Table 3 is from `http://cdsarc.u-strasbg.fr/vizier/ftp/cats/VII/199/ReadMe`
- Office hours: BDF today 3-4pm, or by appointment
  TA: tomorrow 1:30-3:00pm

Last time: ingredients of radiative transfer

- free space *Q: meaning? I_ν result? significance?*
- emission *Q: how quantified? example?*
- absorption *Q: how quantified? example?*

free space:

no emission, absorption, scattering $\rightarrow$ $I_{\nu_1} = I_{\nu_2}$

$$\frac{dI_\nu}{ds} \stackrel{\text{free}}{=} 0 \tag{1}$$

this means: when viewing an object across free space, the *intensity of the object is constant regardless of distance to the object!*

$\Rightarrow$ **conservation of surface brightness**

emission: **emission coefficient** is

$$j_\nu = \frac{d\mathcal{E}_{\text{emit}}}{dV\ dt\ d\Omega\ d\nu} \tag{2}$$

and the *intensity change* is

$$dI_\nu \stackrel{\text{sources}}{=} j_\nu\ ds \tag{3}$$

absorption: *modeled* as:

$$dI_\nu = -\alpha_\nu \; I_\nu \; ds \tag{4}$$

in terms of absorption cross section σ_ν or opacity κ_ν

$$dI_\nu \overset{\mathrm{abs}}{=} -n_{\mathrm{a}} \; \sigma_\nu \; I_\nu \; ds = \rho \; \kappa_\nu \; I_\nu \; ds \tag{5}$$

which has the expected form, with

$$\alpha_\nu = n_{\mathrm{a}} \; \sigma_\nu \tag{6}$$

note that absorption depends on
- *microphysics* via the cross section σ_ν
- *astrophysics* via density n_{abs} of scatterers

often, write $\alpha_\nu = \rho\kappa_\nu$,
defines **opacity** $\kappa_\nu = (n/\rho)\sigma_\nu \equiv \sigma_\nu/m$
with $m = \rho/n$ the mean mass per absorber

Q: so what determines σ_ν? e.g., for electrons?

Cross Sections

Note that the absorption **cross section** σ_ν is
and *effective* area presented by absorber

for "billiard balls" = neutral, opaque, macroscopic objects
this is the same as the geometric size
but generally, cross section is *unrelated to geometric size*
e.g., electrons are point particles (?) but still scatter light

- so *generalize* our ideas so that
  $dI_\nu = -n_{\rm a}\ \sigma_\nu\ ds$ *defines* the cross section
- determined by the details of light-matter interactions
- can be–and usually is!–frequency dependent
- differ according to physical process
  the study of which will be the bulk of this course!

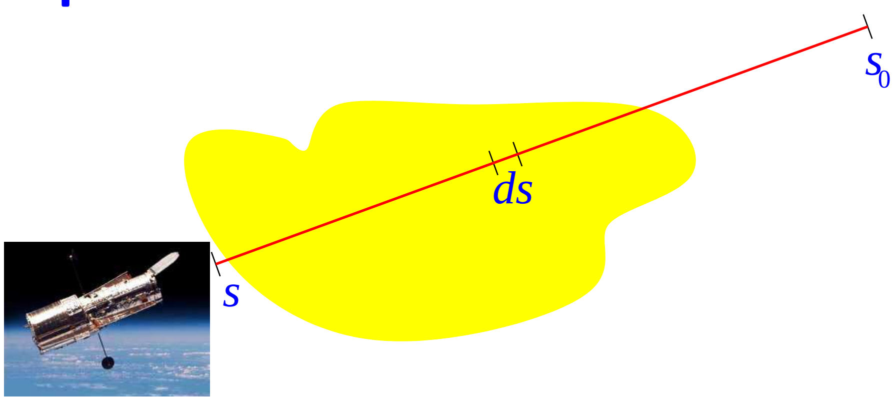
Note: "absorption" here is anything removing energy from beam
$\rightarrow$ can be true absorption, but also scattering

The Equation of Radiative Transfer

Now combine effects of sources and sinks
that change intensity as light propagates

$$\boxed{\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + j_\nu} \tag{7}$$

equation of radiative transfer



- fundamental equation in this course
- *sources* parameterized via j_ν
- *sinks* parameterized via $\alpha_\nu = n_{\rm a}\,\sigma_\nu = \rho\kappa_\nu$

Transfer Equation: Limiting Cases

equation of radiative transfer:

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + j_\nu \tag{8}$$

Sources but no Sinks

if sources exist but there are no sinks: $\alpha_\nu = 0$

$$\frac{dI_\nu}{ds} = j_\nu \tag{9}$$

solve along path starting at sightline distance s_0:

$$I_\nu(s) = I_\nu(s_0) + \int_{s_0}^{s} j_\nu \, ds' \tag{10}$$

- the *increment* in intensity is due to integral of sources *along sightline*
- for $j_\nu \to 0$: free space case and $I_\nu(s) = I_\nu(s_0)$: recover surface brightness conservation!

Special Case: Sinks but no Sources

if absorption only, no sources: $j_\nu = 0$

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu \tag{11}$$

and so on a sightline from s_0 to s

$$I_\nu(s) = I_\nu(s_0)\ e^{-\int_{s_0}^{s} \alpha_\nu\ ds'} \tag{12}$$

- intensity *decrement* is *exponential*!
- exponent depends on line integral of absorption coefficient

useful to define **optical depth** via $d\tau_\nu \equiv \alpha_\nu\ ds$

$$\tau_\nu(s) = \int_{s_0}^{s} \alpha_\nu\ ds' \tag{13}$$

and thus *for absorption only* $I_\nu(s) = I_\nu(s_0) e^{-\tau_\nu(s)}$

Mean Free Path

Average optical depth is

$$\langle \tau_\nu \rangle = \frac{\int \tau_\nu e^{-\tau_\nu} d\tau_\nu}{\int e^{-\tau_\nu} d\tau_\nu} = 1$$

for constant density $n_{\rm a}$, this occurs
at the **mean free path**

$$\ell_{\rm mfp,\nu} = \frac{1}{n_{\rm a}\ \sigma_\nu}$$

average distance between collisions

similarly *mean free time* between collisions

∞

$$\tau_\nu = \frac{\ell_{\rm mfp,\nu}}{c} \tag{14}$$

where we used $v = c$ for all photons

Optical Depth

optical depth, in terms of cross section

$$\tau_\nu(s) = \int_{s_0}^{s} n_{\mathrm{a}}\ \sigma_\nu\ ds' = \int_{s_0}^{s} \frac{ds'}{\ell_{\mathrm{mfp},\nu}} \tag{15}$$

$$= \text{number of mean free paths} \tag{16}$$

optical depth counts mean free paths along sightline
i.e., typical number of absorption events

Limiting cases:

• $\tau_\nu \ll 1$: **optically thin**
absorption unlikely → transparent

• $\tau_\nu \gg 1$: **optically thick**
absorption overwhelmingly likely → opaque

Column Density

Note "separation of variables" in optical depth

$$\tau_\nu(s) \;=\; \underbrace{\sigma_\nu}_{\text{microphysics}} \; \underbrace{\int_{s_0}^{s} n_{\rm a}(s') \; ds'}_{\text{astrophysics}} \tag{17}$$

From observations, can (sometimes) infer τ_ν *Q: how?*
but cross section σ_ν fixed by absorption microphysics
i.e., by theory and/or lab data

absorber astrophysics controlled by **column density**

$$N_a(s) \equiv \int_{s_0}^{s} n_{\rm a}(s') \; ds' \tag{18}$$

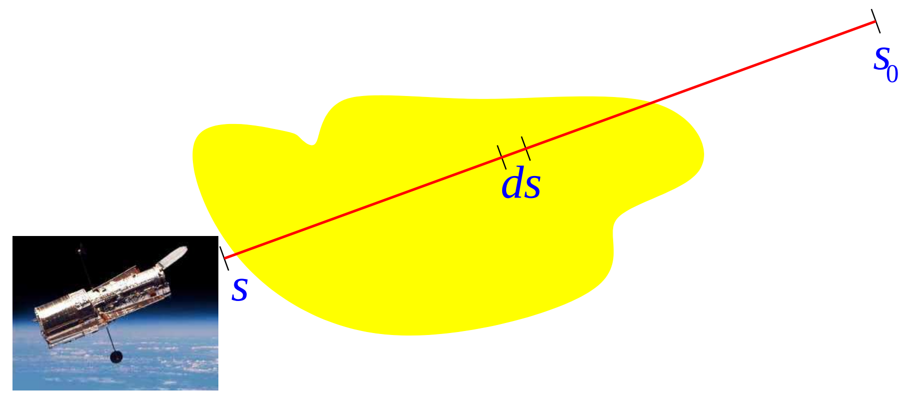
line integral of number density over entire line of sight s
cgs units $[N_{\rm a}] = [\mathrm{cm}^{-2}]$

Q: what does column density represent physically?

column density

$$N_a(s) \equiv \int_{s_0}^{s} n_{\rm a}\; ds'$$

so $\tau_\nu = \sigma_\nu N_{\rm a}$



- column density is projection of 3-D absorber density onto 2-D sky, "collapsing" the sightline
  "cosmic roadkill"

- if source is a slab $\perp$ to sightline, then $N_{\rm a}$ is surface density

- if source is multiple slabs $\perp$ to sightline, then $N_{\rm a}$ sums surface density of all slabs

Q: from $N_{\rm a}$, how to recover 3-D density n_a?

Radiation Transfer Equation, Formal Solution

equation of transfer

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + j_\nu \tag{19}$$

divide by α_ν and rewrite
in terms of optical depth $d\tau_\nu = \alpha_\nu ds$

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu \tag{20}$$

with the **source function**

$$\boxed{S_\nu = \frac{j_\nu}{\alpha_\nu} = \frac{j_\nu}{n_\mathrm{a}\sigma_\nu}} \tag{21}$$

Q: source function dimensions?

Source Function

$S_\nu = j_\nu/\alpha_\nu$ has dimensions of surface brightness
What does it represent physically?

consider the case where the *same* matter
is responsible for both emission and absorption; then:

- $\alpha_\nu = n\sigma_\nu$, with n the particle number density
- $j_\nu = n\, dL_\nu/d\Omega$, with $dL_\nu/d\Omega$ the specific power emitted *per particle* and per solid angle

and thus we have

$$S_\nu = \frac{dL_\nu/d\Omega}{\sigma_\nu} \tag{22}$$

specific power per unit effective area and solid angle
$\rightarrow$ effective surface brightness of each particle!

spoiler alert: S_ν encodes emission vs absorption relation
ultimately set by quantum mechanical symmetries
e.g., time reversal invariance, "detailed balance"

Radiative Transfer Equation: Formal Solution

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu \tag{23}$$

If emission *independent* of I_ν (*not* always true! *Q: why?*)
Then can formally solve

Write $I_\nu = \Phi_\nu e^{-\tau_\nu}$, i.e., use *integrating factor* $e^{-\tau_\nu}$, so

$$\frac{d(\Phi_\nu e^{-\tau_\nu})}{d\tau_\nu} = e^{-\tau_\nu}\frac{d\Phi_\nu}{d\tau_\nu} - \Phi_\nu e^{-\tau_\nu} \tag{24}$$

$$= -\Phi_\nu e^{-\tau_\nu} + S_\nu \tag{25}$$

and so we have

$$\frac{d\Phi_\nu}{d\tau_\nu} = e^{+\tau_\nu} S_\nu(\tau_\nu) \tag{26}$$

and thus

$$\Phi_\nu(s) = \Phi_\nu(0) + \int_0^{\tau_\nu} e^{\tau'_\nu}\ S_\nu(\tau'_\nu)\ d\tau'_\nu \tag{27}$$

$$\Phi_\nu(s) = \Phi_\nu(0) + \int_0^{\tau_\nu(s)} e^{\tau'_\nu} \; S_\nu(\tau'_\nu) \; d\tau'_\nu \qquad (28)$$

and then

$$I_\nu(s) \;=\; \Phi_\nu(s) \; e^{-\tau_\nu(s)} \qquad (29)$$

$$\;=\; I_\nu(0) \; e^{-\tau_\nu(s)} + \int_0^{\tau_\nu(s)} e^{-[\tau_\nu(s)-\tau'_\nu]} \; S_\nu(\tau'_\nu) \; d\tau'_\nu \qquad (30)$$

in terms of original variables

$$I_\nu(s) = I_\nu(0) e^{-[\tau_\nu(s)-\tau_\nu(s_0)]} + \int_{s_0}^{s} e^{-[\tau_\nu(s)-\tau_\nu(s')]} \; j_\nu(\tau'_\nu) \; ds'$$

Q: what strikes you about these solutions?

Formal solution to transfer equation:

$$I_\nu(s) = I_\nu(0)\ e^{-\tau_\nu(s)} + \int_0^{\tau_\nu(s)} e^{-[\tau_\nu(s)-\tau'_\nu]}\ S_\nu(\tau'_\nu)\ d\tau'_\nu \qquad (31)$$

in terms of original variables

$$I_\nu(s) = I_\nu(0) e^{-[\tau_\nu(s)-\tau_\nu(s_0)]} + \int_{s_0}^{s} e^{-[\tau_\nu(s)-\tau_\nu(s')]}\ j_\nu(\tau'_\nu)\ ds'$$

- initial intensity degraded by absorption

- added intensity depends on sources along column
  but optical depth weights against sources with $\tau_\nu \gtrsim 1$

Formal Solution: Special Cases

For spatially *constant* source function $S_\nu = j_\nu/\alpha_\nu$:

$$I_\nu(s) = e^{-\tau_\nu(s)} I_\nu(0) + S_\nu \int_0^{\tau_\nu(s)} e^{-[\tau_\nu(s)-\tau'_\nu]} \, d\tau'_\nu \qquad (32)$$

$$= e^{-\tau_\nu(s)} I_\nu(0) + \left(1 - e^{-\tau_\nu(s)}\right) S_\nu \qquad (33)$$

- optically thin: $\tau_\nu \ll 1$
  $I_\nu \approx (1-\tau_\nu) I_\nu(0) + \tau_\nu S_\nu$
- optically thick: $\tau_\nu \gg 1$
  $I_\nu \to S_\nu$
  $\to$ optically thick intensity is source function!

what's going on? rewrite:

$$\frac{dI_\nu}{ds} = -\frac{1}{\ell_{\mathrm{mfp},\nu}} (I_\nu - S_\nu) \qquad (34)$$

Q: what happens if $I_\nu < S_\nu$? if $I_\nu > S_\nu$?
Q: lesson? characteristic scales?

Radiation Transfer as Relaxation

$$\frac{dI_\nu}{ds} = -\frac{1}{\ell_{\rm mfp,\nu}} (I_\nu - S_\nu) \tag{35}$$

- if $I_\nu < S_\nu$, then $dI_\nu/ds > 0$:
  $\rightarrow$ intensity *increases* along path
- if $I_\nu > S_\nu$, intensity *decreases*

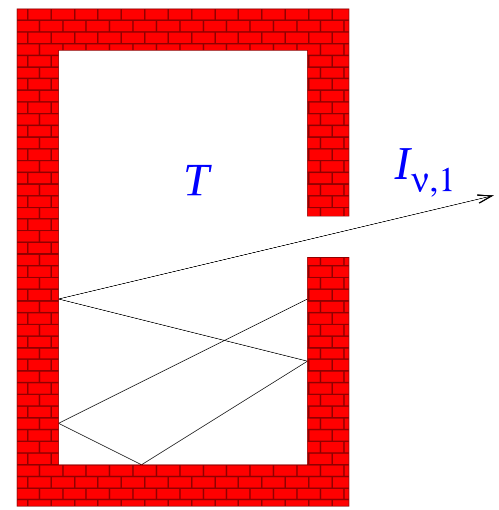
equation is "*self regulating*!"
I_ν "*relaxes*" to "attractor" S_ν

and characteristic lengthscale for relaxation is mean free path!
recall $S_\nu = \ell_{\rm mfp,\nu} j_\nu \approx$: this is "source-only" result
for sightline pathlength $s = \ell_{\rm mfp,\nu}$

Blackbody Radiation

Radiation and Thermodynamics

consider an enclosure ("*box 1*")
in *thermodynamic equilibrium* at temperature T



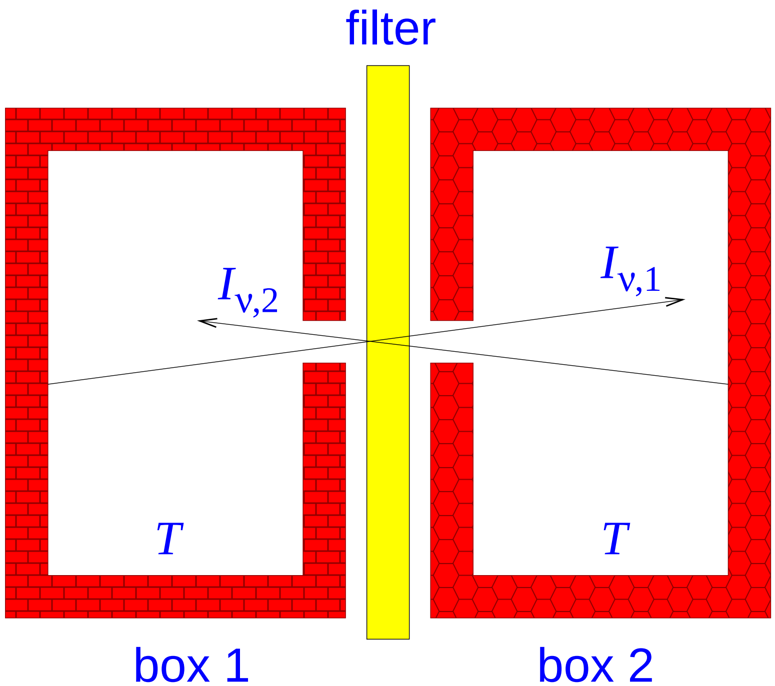
box 1

the matter in box 1

- is in random thermal motion
- will absorb and emit radiation
  details of which depends on
  the details of box material and geometry
- but equilibrium
  $\rightarrow$ radiation field in box doesn't change

open little hole: escaping radiation has intensity $I_{\nu,1}$

now add another enclosure ("box 2"), also at temperature T
but made of *different material*



separate boxes by *filter passing only frequency* ν
radiation from each box incident on screen
Q: imagine $I_{\nu,1} > I_{\nu,2}$*; what happens?*
Q: lesson?

Blackbody Radiation

if both boxes at *same* T $\Rightarrow$ *no net energy transfer*
but this requires $I_{\nu,1} = I_{\nu,2}$ and so the radiation is:

- independent of the composition of the box
- a universal function of T
- **blackbody radiation** with intensity $B_\nu(T)$

Spoiler alert (useful for PS1): blackbody radiation

$$B_\nu(T) = \frac{2h}{c^2} \frac{\nu^3}{e^{h\nu/kT} - 1} \tag{36}$$

with $h =$ Planck's constant, $k =$ Boltzmann's constant

in wavelength space

$$B_\lambda(T) = 2hc^2 \frac{\lambda^{-5}}{e^{hc/\lambda kT} - 1} \tag{37}$$

blackbody integrated intensity:

$$B(T) = \int B_\nu(T)\ d\nu = \int B_\lambda(T)\ d\lambda \tag{38}$$

$$= \frac{2\pi^4}{15}\frac{k^4 T^4}{c^3 h^3} = \frac{\sigma}{\pi} T^4 = \frac{c}{4\pi} a T^4 \tag{39}$$

blackbody flux

$$F_\nu(T) = \pi B_\nu(T) = \frac{2\pi h}{c^2}\frac{\nu^3}{e^{h\nu/kT} - 1} \tag{40}$$

$$F(T) = \pi B(T) \equiv \sigma T^4 = \frac{2\pi^5}{15}\frac{k^4 T^4}{c^2 h^3} \tag{41}$$

defines *Stefan-Boltzmann constant*

$$\sigma = \frac{2\pi^5}{15}\frac{k^4}{c^2 h^3} = 5.670 \times 10^{-5}\ \mathrm{erg\ cm^{-2}\ s^{-1}\ K^{-4}} \tag{42}$$

Q: to order of magnitude: integrated number density?

Note: *blackbody quantities determined entirely by* T
no adjustable parameters!

mean number density: dimensions $[n] = [\text{length}^{-3}]$
can only depend on T, and physical constants h, c, k
can form only one length: $[hc/kT] = [\text{length}]$
$\rightarrow$ expect $n \sim (hc/kT)^3$

photon number density

$$n_\nu(T) = \frac{4\pi B_\nu(T)}{hc\nu} = \frac{8\pi}{c^3} \frac{\nu^2}{e^{h\nu/kT} - 1} \quad (43)$$

$$n(T) = 16\pi\zeta(3) \left(\frac{kT}{hc}\right)^3 \quad (44)$$

where $\zeta(3) = 1 + 1/2^3 + 1/3^3 + 1/4^3 + \cdots = 1.2020569\ldots$

Q: implications?

blackbody photon number density

$$n(T) = 16\pi\zeta(3)\left(\frac{kT}{hc}\right)^3 \tag{45}$$

i.e., $n \propto T^3$

So if temperatures changes, photon number changes

blackbody photon number is not conserved

photons massless $\rightarrow$ can always make more!

if heat up, photon number increases

and spectrum relaxes to blackbody form

blackbody energy density?

to order of magnitude, expect $u \sim nkT \sim (kT)^4/(hc)^3$

integrated energy density

$$u_\nu(T) = \frac{4\pi B_\nu(T)}{c} = \frac{8\pi h}{c^3}\frac{\nu^3}{e^{h\nu/kT}-1} \quad (46)$$

$$u(T) = \frac{4\pi B(T)}{c} = \frac{8\pi^5}{15}\frac{k^4T^4}{c^3h^3} \quad (47)$$

$$\equiv aT^4 = \frac{4\sigma}{c}T^4 \quad (48)$$

defines *Stefan-Boltzmann radiation density constant* $a = 4\sigma/c$

mean photon energy:
only one way to form an energy
$\rightarrow$ expect $\langle E\rangle \sim kT$

exact result:

$$\langle E\rangle \equiv \frac{u(T)}{n(T)} \quad (49)$$

$$= \frac{\pi^4}{30\zeta(3)}kT = 2.701\,kT \quad (50)$$

Director's Cut Extras

Radiative Forces

generalize our definition of flux:
energy flux in direction $\hat{n}$ is

$$\vec{F}_\nu = \int I_\nu \ \hat{n} \ d\Omega \tag{51}$$

recovers old result if we take $\hat{z} \cdot \vec{F}_\nu$

each photon has momentum E/c, and so
momentum per unit area and pathlength
absorbed by medium with absorption coefficient α_ν:

$$\vec{\mathcal{F}} = \frac{d\vec{p}}{dt \ dA \ ds} = \frac{1}{c} \int \alpha_\nu \ \vec{F}_\nu \ d\nu \tag{52}$$

but $dA \ ds = dV$, and $d\vec{p}/dt$ is force,
so $\vec{\mathcal{F}}$ is the **force density**
i.e., force per unit volume, on absorbing matter

force per unit mass is

$$\vec{f} = \frac{\vec{\mathcal{F}}}{\rho} = \frac{1}{c} \int \kappa_\nu \; \vec{F}_\nu \; d\nu \tag{53}$$

Note: we have accounted only force due to *absoption* of radiation

What about *emission?*

If emission is isotropic, no net force
if not, must include this as a separate term