Astro 501: Radiative Processes
Lecture 5
Jan 25, 2013

Announcements:

- **Problem Set 1** due **now**
- **Problem Set 2** available, due at start of class next Friday

Last time: the glorious equation of radiation transfer

Q: what is it?

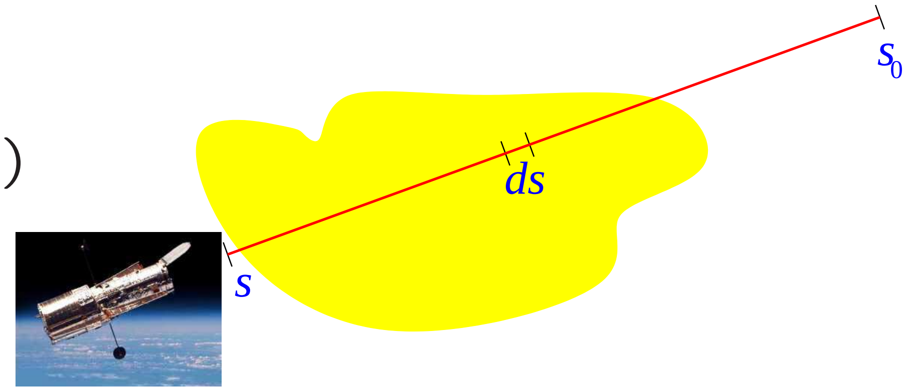
Q: what is optical depth? column density?

Q: what is source function? why is it important?

equation of radiation transfer

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + j_\nu = -\alpha_\nu (I_\nu - S_\nu)$$

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu$$



with source function

$$S_\nu = \frac{j_\nu}{\alpha_n u} \tag{1}$$

and optical depth $d\tau_\nu = \alpha_\nu \; ds$, so that

$$\tau_\nu = \int_{s_0}^{s} \alpha_\nu \; ds = \sigma_\nu \; N_{\rm a} \tag{2}$$

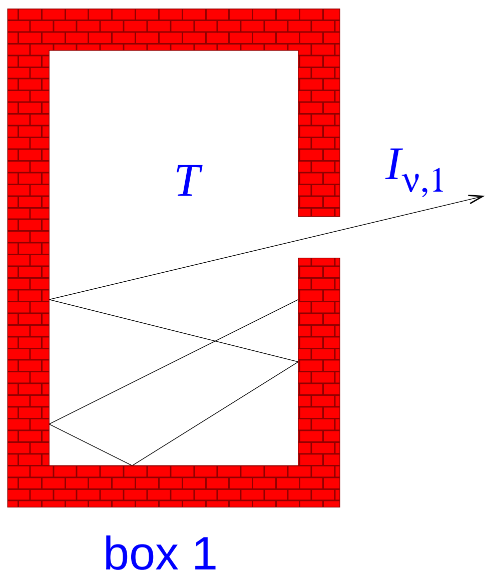
with column density

$$N_{\rm a} = \int_{s_0}^{s} n_{\rm a} \; ds \tag{3}$$

Blackbody Radiation

Radiation and Thermodynamics

consider an enclosure ("*box 1*")
in *thermodynamic equilibrium* at temperature T

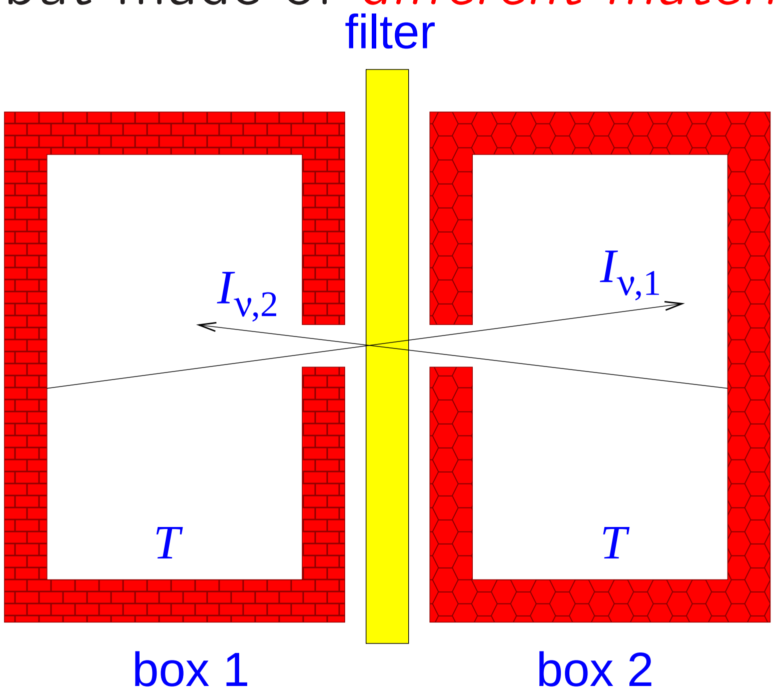


the matter in box 1

- is in random thermal motion
- will absorb and emit radiation
  details of which depends on
  the details of box material and geometry
- but equilibrium
  $\longrightarrow$ radiation field in box doesn't change

open little hole: escaping radiation has intensity $I_{\nu,1}$

now add another enclosure ("box 2"), also at temperature T
but made of *different material*



separate boxes by *filter passing only frequency* ν
radiation from each box incident on screen

Q: imagine $I_{\nu,1} > I_{\nu,2}$*; what happens?*

Q: lesson?

Q: how would $I_{\nu,1}$ *change if we increased the box volume but kept it at* T*?*

Blackbody Radiation

if both boxes at *same T* $\Rightarrow$ *no net energy transfer*
but this requires $I_{\nu,1} = I_{\nu,2}$ and so the radiation is:

- independent of the composition of the box
- a universal function of T
- isotropic *Q: why?*
- **blackbody radiation** with intensity $B_\nu(T)$

Implications:

- $B_\nu(T)$ and thus $B(T)$ depends only on T, not on cavity volume V or composition
- thus blackbody energy density $u(T) = 4\pi B(T)/c$ also depends only on T, not on V
- thus in volume V, photon energy is $U = u(T)\,V$
- and pressure is $P(T) = u(T)/3$, also independent of V

Lesson: radiation has energy, exchanges it with environment
$\rightarrow$ *radiation can be treated thermodynamically*

Thermodynamics Recap

First Law of Thermodynamics: heat is work!
adding *heat energy* dQ to system changes
system *energy* U and/or *pressure* P:

$$dQ = dU + pdV \tag{4}$$

Second Law of Thermodynamics: heat is entropy!

$$T\ dS = dQ \tag{5}$$

together

$$T\ dS = dU + P\,dV \tag{6}$$

and thus entropy $S = S(T, V)$ obeys

$$dS = \frac{dU}{T} + \frac{P}{T} dV \tag{7}$$

entropy $S = S(T, V)$ obeys

$$dS = \frac{dU}{T} + \frac{P}{T}dV \tag{8}$$

and thus we have

$$\partial_T S = \frac{\partial_T U}{T} \tag{9}$$

$$\partial_V S = \frac{\partial_V U + P}{T} \tag{10}$$

which means

$$\partial_V \partial_T S = \frac{\partial_V \partial_T U}{T} \tag{11}$$

$$\partial_T \partial_V S = \frac{\partial_T \partial_V U}{T} - \frac{\partial_V U}{T^2} + \partial_T \left(\frac{P}{T}\right) \tag{12}$$

but mix partial derivatives equal, e.g., $\partial_V \partial_T S = \partial_T \partial_V S$,
and note that $\partial_V U|_T = u$ energy density, so

$$u = T^2 \, \partial_T \left(\frac{P}{T}\right) \tag{13}$$

Radiation Thermodynamics

general thermodynamic considerations give:

$$u = T^2 \, \partial_T \left(\frac{P}{T} \right) \tag{14}$$

now specialize to *radiation:* $P = P(T) = u(T)/3$

$$T \frac{d}{dT} \left(\frac{u}{T} \right) = 3 \frac{u}{T} \tag{15}$$

which gives

$$\begin{aligned} \frac{d(u/T)}{u/T} &= 3 \, \frac{dT}{T} && (16) \\ \ln \left(\frac{u}{T} \right) &= 3 \ln(T) + \ln(a) && (17) \\ u(T) &= a \, T^4 && (18) \end{aligned}$$

radiation energy density

$$u(T) = a\ T^4 \tag{19}$$

- $u(T) \propto T^4$: strong T dependence!
- implies $B(T) = ac/4\pi\ T^4$,
  and $F(T) = \pi B(T) = ac/4\ T^4$
- a is the "radiation constant"
  value not determined by thermodynamics alone

Note: *blackbody quantities fixed entirely by* T
no adjustable parameters!

Radiation Entropy

Using $U = aT^4 V$ and $P = u/3$, can solve for **radiation entropy**

$$S_{\text{rad}} = \frac{4}{3} a T^3 \; V \tag{20}$$

and thus *entropy density* $s_{\text{rad}}(T) = S/V = 4/3 \; aT^3$

if entropy constant in a parcel of radiation
$\longrightarrow$ *adiabatic* process:

$$T_{\text{adiabat}} \propto V^{-1/3} \tag{21}$$

$$P_{\text{adiabat}} \propto T^4_{\text{adiabat}} \propto V^{-4/3} \tag{22}$$

writing $P \propto V^{-\gamma}$, we have
an *adiabatic index* $\gamma_{\text{rad}} = 4/3$

Q: but how do we get the constant a?

Gossip Break: Chandra Story

The Quantum Mechanics of Blackbody Radiation

to have deeper understanding of radiation thermodynamics
and to find radiation constant a
need to study radiation in more detail
$\rightarrow$ need physical picture of radiation

can try classical description: radiation as EM waves
different frequencies ("modes") all thermally excited
$\rightarrow$ gives somewhat wrong answers, e.g., $u(T) = 8\pi\, kT/c^3 \int_0^\infty \nu^2\, d\nu \rightarrow \infty$
"ultraviolet catastrophe"

Historically, this disaster drove Planck & Einstein to a new
microscopic picture of quanta: photons
$\rightarrow$ of course this gives correct blackbody description
in a *statistical mechanics* description of photons

Statistical Mechanics in a Nutshell

classically, **phase space** $(\vec{x}, \vec{p})$
completely describes particle state

Q: phase space lifestyle of single classical 1-D free body?
of single 1-D harmonic oscillator?
Q: a swarm of free bodies? oscillators?

but quantum mechanics $\rightarrow$ uncertainty $\Delta x \Delta p \geq \hbar/2$

semi-classically:
can show that a quantum particle must occupy
a *minimum* phase space "volume"
$(dx\ dp_x)(dy\ dp_y)(dz\ dp_z) = h^3 = (2\pi\hbar)^3$
per quantum state of fixed $\vec{p}$

Distribution Function

define "occupation number" or "distribution function" $f(\vec{x}, \vec{p})$:
number of particles in each phase space "cell"
Q: f range for fermions? bosons?
Q: what is f for one classical particle? many classical particles?

Given distribution function, total number of particles is

$$dN = g f(\vec{x}, \vec{p}) \; \frac{d^3\vec{x} \; d^3\vec{p}}{h^3} \tag{23}$$

where g is # internal (spin/helicity) states:
Q: $g(e^-)$? $g(\gamma)$? $g(p)$?

particle phase space occupation f determines bulk properties
Q: how? Hint–what's # particles per unit spatial volume?

Fermions: $0 \leq f \leq 1$ (Pauli)
Bosons: $f \geq 0$ $g(e^-) = 2s(e^-) + 1 = 2$ electron, same for p
$g(\gamma) = 2$ *(polarizations) photon*

Particle phase space occupation f determines bulk properties

Number density

$$n(\vec{x}) = \frac{d^3N}{d^3x} = \frac{g}{h^3} \int d^3\vec{p}\ f(\vec{p}, \vec{x}) \tag{24}$$

Q: this expressions is general–specialize to photons?

for photons $E = cp = h\nu$
so $d^3p = p^2\, dp\, d\Omega = h^3/c^3\ \nu^2\ d\nu\ d\Omega$

photon number density is thus

$$dn = \frac{2}{c^3}\nu^2\ f(\nu)\ d\nu\ d\Omega \tag{25}$$

and thus we have

$$\frac{dn_\nu}{d\Omega} = \frac{dn}{d\nu\ d\Omega} = \frac{2}{c^3}\nu^2\ f(\nu) \tag{26}$$

thus f *gives a general, fundamental description of photon fields*
the challenge is to find the physics that determines f
$\longrightarrow$ spoiler alert: you have already seen a version of it!
but will see it again as the Boltzmann equation!

Note: distribution function $f(\nu)$ and specific intensity I_ν
are *equivalent* and *interchangeable descriptions*
Q: why? how do we get I_ν *from* $f(\nu)$*?*

Distribution Function and Observables

distribution function $f(\nu)$ is related to photon number via

$$\frac{dn_\nu}{d\Omega} = \frac{dN}{dV\ d\nu\ d\Omega} = \frac{2}{c^3}\nu^2\ f(\nu) \tag{27}$$

but we found that photon specific intensity is related to specific number density via

$$I_\nu = c\ h\nu\ \frac{dn_\nu}{d\Omega} \tag{28}$$

but this means that the two are related via

$$\boxed{I_\nu = \frac{2h}{c^2}\nu^3\ f(\nu)} \tag{29}$$

Equilibrium Occupation Numbers

So far, totally general description of photon fields
no assumption of thermodynamic equilibrium

in thermodynamical equilibrium at T, the distribution function
is also the *occupation number*
i.e., average *number* of photons with freq ν

$$f(\nu, T) = \frac{1}{e^{h\nu/kT} - 1} \tag{30}$$

see derivation in today's Director's Cut Extras

Q: at fixed T, for which ν is f large? small?
Q: sketch of $f(\nu)$?
Q: what does this all mean physically?
Q: when is f zero?
Q: in which regime do we expect classical behavior? quantum?

Blackbody Radiation Properties

Using the blackbody distribution function, we define

$$B_\nu(T) \equiv I_\nu(T) = \frac{2h}{c^2}\nu^3 \; f(\nu, T) \tag{31}$$

and thus we have

$$B_\nu(T) = \frac{2h}{c^2}\frac{\nu^3}{e^{h\nu/kT} - 1} \tag{32}$$

with $h =$ Planck's constant, $k =$ Boltzmann's constant

in wavelength space

$$B_\lambda(T) = 2hc^2\frac{\lambda^{-5}}{e^{hc/\lambda kT} - 1} \tag{33}$$

Director's Cut Extras

Blackbody Photon Occupation Number

at a fixed temperature T and frequency ν
we want the distribution function f, i.e., the occupation number
i.e., the average number of photons with frequency ν

Boltzmann: probability of having state n of energy E_n
proportional to $p_n = e^{-E_n/kT}$

Planck: n photons have $E_n = h\nu$, so $p_n = e^{-nx}$
with $x = h\nu/kT$

So average number is

$$f = \langle n \rangle = \frac{\sum_n n p_n}{\sum_n p_n} = \frac{\sum_n n e^{-nx}}{\sum_n e^{-nx}} \quad (34)$$

note that $\sum_n n e^{-nx} = -\partial_x \sum_n e^{-nx}$, so

$$f = -\partial_x \ln \left(\sum_n e^{-nx} \right) \tag{35}$$

but geometric series has sum

$$\sum_n e^{-nx} = \sum_n (e^{-x})^n = \frac{1}{1 - e^{-x}} \tag{36}$$

and thus

$$f = -\partial_x \ln \frac{1}{1 - e^{-x}} = \partial_x \ln(1 - e^{-x}) \tag{37}$$

$$= \frac{e^{-x}}{1 - e^{-x}} \tag{38}$$

which gives

$$f(\nu, T) = \frac{1}{e^{h\nu/kT} - 1} \tag{39}$$

which was to be shewn