Astro 501: Radiative Processes
Lecture 6
Jan 28, 2013

Announcements:

- **Problem Set 2** sue at start of class Friday

Last time:

- radiation thermodynamics
  Q: blackbody integrated energy density $u(T)$*?*
- statistical mechanics
  distribution function f *Q: physical meaning? units?*
  Q: connection between f *and* I_ν*?*
  Q: f *for blackbody?*

Blackbody Radiation Properties

Using the blackbody distribution function, we define

$$B_\nu(T) \equiv I_\nu(T) = \frac{2h}{c^2}\nu^3 \; f(\nu, T) \qquad (1)$$

and because $f(\nu, T) = 1/(e^{h\nu/kT} - 1)$, we have

$$B_\nu(T) = \frac{2h}{c^2}\frac{\nu^3}{e^{h\nu/kT} - 1} \qquad (2)$$

with $h =$ Planck's constant, $k =$ Boltzmann's constant

in wavelength space

$$B_\lambda(T) = 2hc^2\frac{\lambda^{-5}}{e^{hc/\lambda kT} - 1} \qquad (3)$$

blackbody integrated intensity:

$$B(T) = \int B_\nu(T)\ d\nu = \int B_\lambda(T)\ d\lambda \tag{4}$$

$$= \frac{2\pi^4}{15}\frac{k^4T^4}{c^3h^3} = \frac{\sigma}{\pi}T^4 = \frac{c}{4\pi}aT^4 \tag{5}$$

blackbody flux

$$F_\nu(T) = \pi B_\nu(T) = \frac{2\pi h}{c^2}\frac{\nu^3}{e^{h\nu/kT} - 1} \tag{6}$$

$$F(T) = \pi B(T) \equiv \sigma T^4 = \frac{2\pi^5}{15}\frac{k^4T^4}{c^2h^3} \tag{7}$$

defines *Stefan-Boltzmann constant*

$$\sigma = \frac{2\pi^5}{15}\frac{k^4}{c^2h^3} = 5.670 \times 10^{-5}\ \text{erg cm}^{-2}\ \text{s}^{-1}\ \text{K}^{-4} \tag{8}$$

Q: to order of magnitude: integrated number density?

mean number density: dimensions $[n] = [\text{length}^{-3}]$
can only depend on T, and physical constants h, c, k
can form only one length: $[hc/kT] = [\text{length}]$
$\rightarrow$ expect $n \sim (hc/kT)^3$

photon number density

$$n_\nu(T) = \frac{4\pi B_\nu(T)}{hc\nu} = \frac{8\pi}{c^3} \frac{\nu^2}{e^{h\nu/kT} - 1} \quad (9)$$

$$n(T) = 16\pi\zeta(3) \left(\frac{kT}{hc}\right)^3 \quad (10)$$

where $\zeta(3) = 1 + 1/2^3 + 1/3^3 + 1/4^3 + \cdots = 1.2020569\ldots$

Q: implications?

blackbody photon number density

$$n(T) = 16\pi\zeta(3)\left(\frac{kT}{hc}\right)^3 \tag{11}$$

i.e., $n \propto T^3$

So if temperatures changes, photon number changes

blackbody photon number is not conserved

photons massless $\rightarrow$ can always make more!

if heat up, photon number increases

and spectrum relaxes to blackbody form

integrated energy density

$$u_\nu(T) = \frac{4\pi B_\nu(T)}{c} = \frac{8\pi h}{c^3}\frac{\nu^3}{e^{h\nu/kT}-1} \tag{12}$$

$$u(T) = \frac{4\pi B(T)}{c} = \frac{8\pi^5}{15}\frac{k^4T^4}{c^3h^3} \tag{13}$$

$$\equiv aT^4 = \frac{4\sigma}{c}T^4 \tag{14}$$

Stefan-Boltzmann radiation density constant

$$a = \frac{4\sigma}{c} = 7.56 \times 10^{-15}\ \text{erg cm}^{-3}\ \text{K}^{-4} \tag{15}$$

at last!

Blackbody Spectral Properties

at fixed ν, occupation number $\partial_T f(\nu, T) > 0$

$\rightarrow$ more photons, larger f for larger T

$\rightarrow$ more specific intensity, flux, energy density, at larger T

$\rightarrow$ slogan: *"blackbody spectra at different T never cross"*

natural energy scale kT, sets two limits

Rayleigh-Jeans limit $h\nu \ll kT$

occupation number $f(\nu) \rightarrow kT/h\nu \gg 1$

many photons, expect classical behavior

specific intensity $I_\nu = 2h/c^2 \; \nu^3 f \rightarrow 2kT\, \nu^2/c^2$

- $I_\nu \propto \nu^2$: power-law scaling
- h does not appear in I_ν: classical behavior!

Wien limit $h\nu \gg kT$

occupation number $f(\nu) \to e^{-h\nu/kT} \ll 1$

photon starved: thermal bath cannot "pay energy cost"

specific intensity $I_\nu \to 2h\,\nu^3/c^2\ e^{-h\nu/kT}$

• exponentially damped due to quantum effects

`www: CMB Rayleigh-Jeans side vs Wien side`

Brightness Temperature

in the Rayleigh-Jeans limit $h\nu \ll kT$ (high ν, long λ)
we found $I_\nu \to 2kT\,\nu^2/c^2$:
$\to$ specific intensity *linearly proportional* to T

define **brightness temperature** or **antenna temperature**

$$T_{\rm b} \equiv \frac{c^2}{2k}\,\frac{I_\nu}{\nu^2} \tag{16}$$

note: can always define, even for nonthermal radiation
also: in general, depends on ν

Q: in what EM band(s) is $T_{\rm b}$ useful?
Q: why is it useful?
Q: what does it mean for a nonthermal source?
Q: when is it not useful?

`www: famous discovery`
`www: modern` $T_{\rm b}$ `results plotted`

Wien's Displacement Laws

for blackbodies,
specific intensity, and flux, and energy density have

$$I_\nu \propto F_\nu \propto u_\nu \propto \frac{\nu^3}{e^{h\nu/kT} - 1} \tag{17}$$

at fixed T, these *spectra all peak at same frequency*

maximum when $x = h\nu/kT$ satisfies $x = 3(1 - e^{-x})$
$\rightarrow x_{\rm max} = 2.821439\ldots$, which gives

$$\frac{\nu_{\rm max}}{T} = x_{\rm max}\frac{kT}{h} = 5.88 \times 10^{10} \text{ Hz K}^{-1} \tag{18}$$

i.e., $\nu_{\rm max} \propto T$, as expected from dimensional analysis in wavelength space, $I_\lambda \propto \lambda^{-5}/(e^{hc/\lambda kT} - 1)$

maximum when $y = hc/\lambda kT$ satisfies $y = 5(1 - e^{-y})$
$\rightarrow y_{\rm max} = 4.9651\ldots$, which gives

$$\lambda_{\rm max}\, T = \frac{1}{y_{\rm max}} \frac{hc}{k} = 0.290 \text{ cm K} \tag{19}$$

i.e., $\lambda_{\rm max} \propto 1/T$, as expected from dimensional analysis

crucial gotcha: **beware!** $\lambda_{\rm max} \neq c/\nu_{\rm max}$
Q: why?

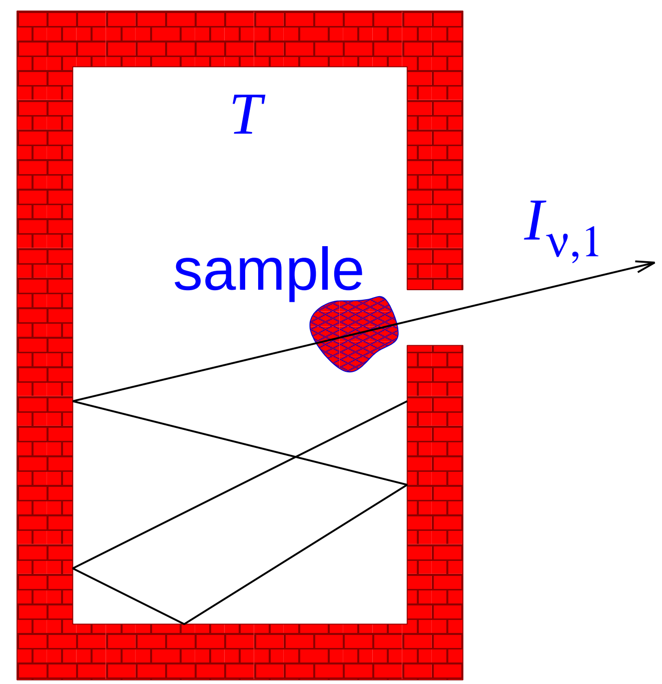
Thermal Radiation Transfer

Consider a cavity in thermodynamic equilibrium

place *small sample of matter* in cavity
along peephole sightline

note: "sample," most generally:

- *not* necessarily large
- *nor* optically thick!
- and might contain free ions, and/or bound states (atoms and molecules)



allow system to come into *thermodynamic equilibrium* at T

12 *Q: effect on I_ν emerging from peephole?*

Q: implications?

matter sample will emit and absorb radiation
according to its detailed composition,
and according to its state at T (i.e., ion, atom, molecule)
$\Rightarrow$ absorption $\alpha_\nu(T)$ and emission $j_\nu(T)$
both characteristic of the matter, *not universal!*

But in cavity+sample system, there is thermal equilibrium
$\rightarrow$ must still be blackbody emitter!
$\rightarrow$ *emitted radiation must have* $I_{\nu,\mathrm{out}} = B_\nu(T)$

Yet radiation *incident* on sample in cavity
already was blackbody: $I_{\nu,\mathrm{in}} = B_\nu(T)$

specific intensity change through sample

$$\frac{dI_\nu}{ds} = -\alpha_\nu (I_\nu - S_\nu) \tag{20}$$

Q: so what condition required for ray to maintain
$I_{\nu,\mathrm{in}} = I_{\nu,\mathrm{out}} = B_\nu(T)$ *through sample?*

Kirchhoff's Law for Thermal Emission

blackbody radiation $B_\nu(T)$ must remain unchanged
when traversing arbitrary matter with temperature T
$\Rightarrow$ demands that *all matter* at T *has source function*

$$S_\nu(T) = B_\nu(T) \tag{21}$$

Kirchhoff's Law for thermal emission

Physically: equilibrium forces *emission–absorption relation*:

$$j_\nu = \alpha_\nu \ B_\nu(T) \tag{22}$$

Consequences:
- a good (poor) emitter is a good (poor) absorber
- *thermal* emission has $S_\nu = B_\nu(T)$ and has

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + B_\nu \tag{23}$$

Q: so when will thermal *source have* blackbody *spectrum?*

thermal source has $S_\nu(T) = B_\nu(T)$

as seen in the PS1 Olber's problem!

and so transfer equation becomes

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + B_\nu$$

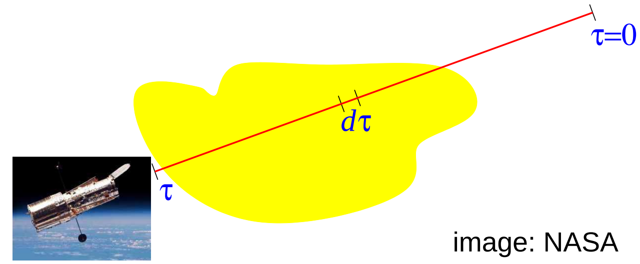


image: NASA

but since $B_\nu(T)$ is homogeneous, isotropic

if no background source, then $I_\nu(0) = 0$, and solution is

$$I_\nu(s) = \left(1 - e^{-\tau_\nu(s)}\right) B_\nu(T) \quad (24)$$

- if *optically thin*, then observe $I_\nu \approx \tau_\nu B_\nu(T) = j_\nu(T)\ \Delta s$
  intensity *not blackbody, and depends on emitter physics*
  `www: Cas A X-ray spectrum (Chandra)` *Q: what is Cas A?*

- if *optically thick*, then $I_\nu \to B_\nu$ as seen in PS1
  intensity relaxes to emitter-independent *blackbody* radiation

Director's Cut Extras

Average Energy per Blackbody Photon

only one way to form an energy
$\rightarrow$ expect $\langle E \rangle \sim kT$; exact result:

$$\langle E \rangle \equiv \frac{u(T)}{n(T)} \quad (25)$$

$$= \frac{\pi^4}{30\zeta(3)} kT = 2.701\, kT \quad (26)$$

c.f. nonrelativistic ideal gas: $\langle E \rangle_{\rm idealgas} = 3/2kT$

note: blackbody radiation has

$$\frac{P}{n\ kT} = \frac{\langle E \rangle}{3} = 0.900 \quad (27)$$

c.f. nonrelativistic ideal gas: $P_{\rm idealgas}/n_{\rm idealgas}kT = 1$

Average Entropy per Blackbody Photon

mean entropy per photon:

entropy has units of Boltzmann's k

$\rightarrow$ expect $\langle S \rangle \sim k$; exact result

$$\langle S \rangle = \frac{s(T)}{n(T)} = frac4u(T)/3Tn(T) = \frac{4}{3}\frac{\langle E \rangle}{T} = 3.601\, k \tag{28}$$

temperature independent!

c.f. nonrelativistic ideal gas: entropy per particle

given by Sackur-Tetrode equation

$$\frac{s_{\rm idealgas}}{n_{\rm idealgas}} = k\left[\frac{5}{2} - \ln\left(\frac{n}{(2\pi mkT/h)^{3/2}}\right)\right] \tag{29}$$

depends on T and density n