Astro 501: Radiative Processes
Lecture 7
Jan 30, 2013

Announcements:

- **Problem Set 2** sue at start of class Friday

Last time: Kirchhoff's law

Q: what is it?

Q: why might it seem like a miracle?

Q: why might it not *seem like a miracle?*
hint–consider emission/absorption if Kirchhoff not true?

Q: what's the difference between a thermal *radiation and* blackbody *radiation?*

Kirchhoff's Law Recap

Kirchhoff: *any* matter in thermal equilibrium at T:

$$S_\nu(T) = B_\nu(T) \tag{1}$$
$$j_\nu(T) = \alpha_\nu(T)\ B_\nu(T) \tag{2}$$

- emission rate related to absorption rate
- good emitters are good absorbers

How does emission "know" to be related to absorption?

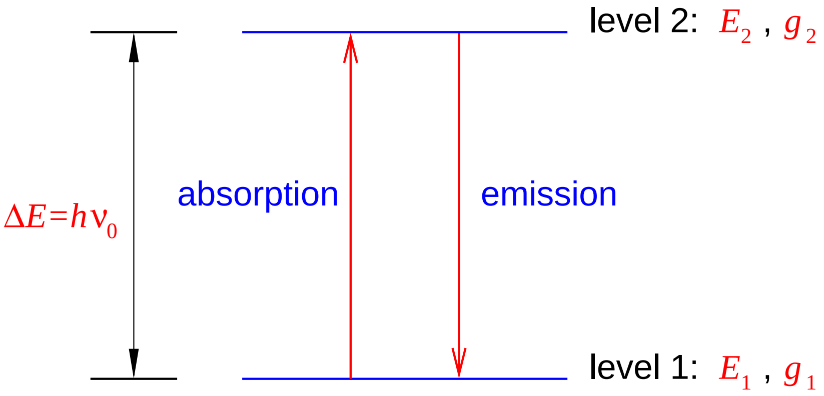
If emission exceeded absorption, matter loses energy
would cool until emission = absorption
condition of equilibrium enforces "detailed balance"

thermal radiation: emitted by *any matter at* T
blackbody radiation: emitted *optically thick* matter at T

Two-Level Systems in Radiative Equilibrium

consider an ensemble of systems (“atoms”) with

- **two discrete energy levels** E_1, E_2
- and degeneracies g_1, g_2
  i.e., a number g_1 of distinct states have energy E_1



level 2: E_2 , g_2

$\Delta E = h\nu_0$

absorption

emission

level 1: E_1 , g_1

in thermodynamic equilibrium at T, emission and absorption exchange energy with photon field

Q: when & why emit? absorb?
Q: connection between emission, absorption rates in ensemble?

Spontaneous Emission

in general, atoms in states with higher energy E_2
will *decay* to lower level E_1
photon of energy $\Delta E = E_2 - E_1 = h\nu_0$ will be emitted

transition can occur without influence of other atoms, photons
spontaneous emission: $X_2 \rightarrow X_1 + h\nu$

spontaneous emission rate per atom is
$E_2 \rightarrow E_1$ *transition rate per atom*:

$$\text{transition probability per unit time per atom} = A_{21} \tag{3}$$

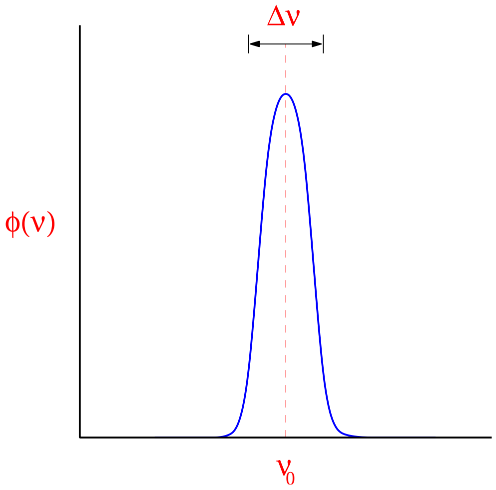
- "**Einstein** A" coefficient
- units $[A_{21}] = [\text{sec}^{-1}]$
- spontaneous: A_{21} independent of T
- but A_{21} *does depend on detailed atom properties*

Absorption

atoms in lower state E_1 only promoted to state E_2
by absorbing a photon of energy ΔE
$h\nu + X_1 \to X_2$

if levels were perfectly sharp, absorb only at $\Delta E = h\nu_0$
but in general, energy levels have *finite width*
i.e., *line energies "smeared out"* by some amount $h\,\Delta nu$
so transitions can be made by photons with frequencies
$\nu_0 - \Delta\nu \ \lesssim \ \nu \ \lesssim \nu_0 + \Delta\nu$

useful to define **line profile function** $\phi(\nu)$
with normalization $\int \phi(\nu)\ d\nu = 1$
e.g., Gaussian, Lorentzian, Voight functions



limiting case of sharp levels $\Delta\nu \to 0$:
$\phi(\nu) \to \delta(\nu - \nu_0)$

absorptions require ambient photons

thus absorption rate per atom *depends on photon field*
and ensemble *average absorption rate*
depends on average intensity

$$\bar{J} \equiv \int \phi(\nu)\ J_\nu\ d\nu \tag{4}$$

limiting case of sharp levels: $\bar{J} \to J_{\nu_0}$

thus write average absorption rate as

$$\text{transition probability per time per atom} \quad = B_{12}\bar{J} \tag{5}$$

- "**Einstein B coefficient**"
- B_{12} is probability per time per intensity
- depends on atom and state details
- but *does not depend on T*

Stimulated Emission

Einstein postulated a new emission mechanism:
driven by photons with transition energy $\nu_0 - \Delta\nu \lesssim \nu \lesssim \nu_0 + \Delta\nu$
$h\nu + X_2 \to X_1 + 2h\nu$

I.e., the presence of transition photons creates "peer pressure"
"encourages" atoms in higher state to make transition
faster than they would spontaneously: **stimulated emission**
plausible? yes–photons interact with and perturb atoms

if stimulated emission exists, should also depend on $\bar{J}$
→ rate per atom is

$$\text{transition probability per time per atom} = B_{21}\bar{J} \qquad (6)$$

- note stimulated emission coefficient B_{21}
  can be *different from* absorption coefficient B_{12}
- if stimulated emission doesn't happen, would find $B_{21} = 0$

The Equilibrium Condition

In thermodynamic equilibrium, the numbers n_1, n_2 of atoms in each state do not change with time
→ total emission rate is equal to absorption rate

$$n_2 A_{21} + n_2 B_{21} \bar{J} = n_1 B_{12} \bar{J} \tag{7}$$

solve for ambient radiation field

$$\bar{J} = \frac{A_{21}/B_{21}}{n_1/n_2 \; B_{12}/B_{21} - 1} \tag{8}$$

in thermodynamic equilibrium, atom state populations follow *Boltzmann distribution*

$$\frac{n_1}{n_2} = \frac{g_1 e^{-E_1/kT}}{g_2 e^{-E_2/kT}} = \frac{g_1}{g_1} e^{(E_2-E_1)/kT} = \frac{g_1}{g_1} e^{h\nu_0/kT} \tag{9}$$

and so

$$\bar{J} = \frac{A_{21}/B_{21}}{g_1/g_2 \; B_{12}/B_{21} \; e^{h\nu_0/kT} - 1} \tag{10}$$

thus we find that in equilibrium,
the mean intensity near ν_0 is

$$\bar{J} = \frac{A_{21}/B_{21}}{g_1/g_2 \; B_{12}/B_{21} \; e^{h\nu_0/kT} - 1} \tag{11}$$

but in equilibrium, and with narrow linewidth,
the mean intensity should be blackbody result:

$$\bar{J} \rightarrow B_\nu(T) = \frac{2h\nu^3/c^2}{e^{h\nu/kT} - 1} \tag{12}$$

Q: what condition(s) must hold to satisfy
both *equations for* any T*?*

because A and both B do not depend on T

the only way to have, at any T,

$$\bar{J} = \frac{2h\nu^3/c^2}{e^{h\nu/kT} - 1} = \frac{A_{21}/B_{12}}{g_1/g_2 \; B_{12}/B_{21} \; e^{h\nu/kT} - 1} \tag{13}$$

is to require two *Einstein relations*

$$A_{21} = \frac{2h \; \nu^3}{c^2} B_{21} \tag{14}$$

$$g_2 B_{21} = g_1 B_{12} \tag{15}$$

- these relations are independent of T:
  hold even without thermal equilibrium!
- $B_{21} \neq 0$: spontaneous emission exists!
  and typically has probability comparable to absorption!
  give it up for Big Al!

Two-Level Systems: Thermal Radiation

Now consider the two-level atom as a radiating system
What are the emission and absorption coefficients?

Emission Coefficient

spontaneous emission rate per atom in state 2: A_{21}
→ rate per volume: $n_2 A_{21}$
→ total power emitted per volume: $h\nu_0 \, n_2 A_{21}$
emission isotropic → power per volume per solid angle:
$h\nu_0 \, n_2 A_{21}/4\pi$ *Q: why?*

but still need *frequency spectrum*
of emitted radiation, i.e., *emission profile*

Q: simplest assumption?

simplest assumption (generally accurate):
→ emission spectrum profile = absorption profile $\phi(\nu)$

and thus energy released in spontaneous emission is

$$d\mathcal{E} = \frac{h\nu_0}{4\pi} n_2 A_{21} \phi(\nu) \ dV \ dt \ d\nu \ d\Omega \tag{16}$$

and thus the emission coefficient is

$$j_\nu = \frac{h\nu_0}{4\pi} n_2 A_{21} \phi(\nu) \tag{17}$$

absorption coefficient

absorption rate per atom in level 1: $B_{12}\bar{J}$

thus energy absorbed is

$$d\mathcal{E} = \frac{h\nu_0}{4\pi} n_1 B_{12} \bar{J} \; dV \; dt \tag{18}$$

but $4\pi\bar{J} = \int d\Omega \int I_\nu \phi(\nu) \; d\nu$, so

$$d\mathcal{E} = \frac{h\nu_0}{4\pi} n_1 B_{12} I_\nu \phi(\nu) \; dV \; dt \; d\Omega \; d\nu \tag{19}$$

recall: path element ds in area dA has volume $dV = ds \; dA$ and so we find absorption coefficient

$$\alpha_{\mathrm{abs},\nu} = \frac{h\nu_0}{4\pi} n_1 B_{12} \phi(\nu) \tag{20}$$

...but we are not done! *Q: because...?*

stimulated emission

tempting to include this as additional emission term

but wait! stimulated emission depends on (average) intensity
→ formally more similar to absorption

formally better to treat stimulated emission as
a *negative absorption* term:

$$\alpha_{\text{stim},\nu} = -\frac{h\nu_0}{4\pi} n_2 B_{21} \phi(\nu) \tag{21}$$

and then *(net) absorption coefficient*

$$\begin{aligned} \alpha_\nu &= \alpha_{\text{abs},\nu} + \alpha_{\text{stim},\nu} && (22) \\ &= \frac{h\nu_0}{4\pi}\phi(\nu)\ (n_1 B_{12} - n_2 B_{21}) && (23) \end{aligned}$$

Two-Level Radiation Transfer

Transfer equation for two-level atom

$$\frac{dI_\nu}{ds} = -\frac{h\nu_0}{4\pi}\phi(\nu)\ \left(n_1 B_{12} - n_2 B_{21}\right) I_\nu + \frac{h\nu_0}{4\pi} n_2 A_{21} \phi(\nu) \tag{24}$$

source function

$$S_\nu = \frac{n_2 A_{21}}{n_1 B_{12} - n_2 B_{21}} \tag{25}$$

Einstein relations give

$$\alpha_\nu = \frac{h\nu_0}{4\pi}\phi(\nu)\ n_1 B_{12} \left(1 - \frac{n_2}{n_1}\frac{g_1}{g_2}\right) \tag{26}$$

$$S_\nu = \frac{2h\nu^3/c^2}{(n_1/n_2)(g_2/g_1) - 1} \tag{27}$$

a generalization of Kirchhoff's laws

these *do not assume thermal equilibrium!*

Q: interesting cases?

Local Thermodynamic Equilibrium

if atom levels are in thermodynamic equilibrium
then we have $n_1/n_2 = (g_1/g_2)e^{h\nu/kT}$ and

$$S_\nu = \frac{2h\nu^3/c^2}{e^{h\nu/kT} - 1} = B_\nu(T) \tag{28}$$

we recover the usual Kirchhoff's law! as we must!

and absorption term becomes

$$\alpha_\nu = \frac{h\nu_0}{4\pi}\phi(\nu)\ n_1 B_{12}\left(1 - e^{-h\nu/kT}\right) \tag{29}$$

i.e., "uncorrected" term minus stimulated emission correction

What if not in thermodynamic equilibrium?
then $n_1/n_1 \neq$ Boltzmann expression
emission is nothermal

Inverted Populations

two-level absorption coefficient is:

$$\alpha_\nu = \frac{h\nu_0}{4\pi}\phi(\nu)\ n_1 B_{12} \left(1 - \frac{n_2}{n_1}\frac{g_1}{g_2}\right) \tag{30}$$

note that the algebraic *sign*
depends on population levels, i.e., on n_2/n_1
normally, lower level more populated: $n_1 > n_2$

If we can arrange or stumble upon a system where

$$\frac{n_1}{g_1} < \frac{n_2}{g_2} \tag{31}$$

i.e., an *inverted population*, then $\alpha_\nu < 0$!

Q: and then what happens to propagating light?
Q: examples?

Q: how might we arrange an inverted population?

Masers

if $\alpha_\nu < 0$, then propagating light has
exponential *increase* in intensity!

stimulated emission causes a “cascade” of photons

in lab: create inverted populations of atoms
use mirrors to “recycle” stimulating photons
→ this is a laser! light amplification by stimulated emission of radiation

in cosmos: inverted populations of molecules
maser: microwave amplification by stimulated emission of radiation

how to create inversion?
need *nonthermal* mechanism to “pump” upper level

Scattering

Pure Scattering

Consider an idealized case with radiation propagating through a medium with "pure scattering," i.e., scattering, but no emission, and no absorption

Recall: intensity in a ray is a directional quantity
i.e., really $I_\nu = I_\nu(\theta, \phi) = I_\nu(\hat{n})$,
with $\hat{n}$ a unit vector toward $I(\theta, \phi)$

in general: scattering will change both
- photon energy
- photon direction

generally, scattering is different for different incident and scattered angles, i.e., anisotropic

this is generally is (very) not trivial to calculate but consider even more special case:

- isotropic scattering
- photon energy unchanged ("coherent scattering")
  good approximation for scattering by non-relativistic e

define scattering coefficient $\alpha_{\rm sc}$ such that
intensity lost to scattering *out* of ray is

$$dI_\nu = -\alpha_{{\rm sc},\nu}\ I_\nu\ ds \tag{32}$$

isotropic scattering → $\alpha_{{\rm sc},\nu}$ same for all directions

Q: what is intensity scattered into *the ray?*

Isotropic Coherent Scattering

intensity scattered *out* of ray $I_\nu(\hat{n}')$
with solid angle $d\Omega'$ is

$$dI_\nu(\hat{n}') = -\alpha_{\mathrm{sc},\nu}\ I_\nu(\hat{n}')\ ds \tag{33}$$

if scattering *isotropic*, the portion *into* $\hat{n}$ is

$$dI_\nu(\hat{n}) = \frac{d\Omega'}{4\pi} \left| dI_\nu(\hat{n}') \right| \tag{34}$$

and so integrating over all possible solid $d\Omega'$ gives

$$dI_\nu(\hat{n}) = \frac{\alpha_{\mathrm{sc},\nu}}{4\pi} \int I_\nu\ d\Omega\ ds = \alpha_\nu\ J_\nu\ ds \tag{35}$$

where J_ν is the angle-averaged intensity and thus for isotropic coherent scattering

$$\frac{dI_\nu(\hat{n})}{ds} = -\alpha_{\mathrm{sc},\nu}\left[I_\nu(\hat{n}) - J_\nu\right] \tag{36}$$

and so the source function is

$$S_\nu = J_\nu \tag{37}$$

and the transfer equation can be written

$$\frac{dI_\nu(\hat{n})}{d\tau_\nu} = -I_\nu(\hat{n}) + J_\nu \tag{38}$$

note that the mean flux $J_\nu = \int I_\nu(\hat{n}')d\Omega'/4\pi$
depends on I_ν field in *all directions*

- scattering couples intensity in different directions
- transfer equation becomes integro-differential equation
  generally very hard to solve!