Astro 501: Radiative Processes
Lecture 8
Feb 1, 2013

Announcements:
- **Problem Set 2** due now
- **Problem Set 3** available, due next Friday

Last time: radiative properties of a two-level system
Q: emission processes? absorption? what do they depend on?

Today: scattering

Scattering

Pure Scattering

Consider an idealized case with radiation propagating through a medium with "*pure scattering*," i.e., scattering, but *no emission*, and *no absorption*

Recall: intensity in a ray is a directional quantity
i.e., really $I_\nu = I_\nu(\theta, \phi) = I_\nu(\hat{n})$,
with $\hat{n}$ a unit vector toward $I(\theta, \phi)$

in general: scattering preserves photon *number*
but *redistributes* both

- photon energy
- photon direction

generally, scattering is different for different incident and scattered angles, i.e., anisotropic

this is generally is (very) non-trivial to calculate but consider even more special case:

- *isotropic* scattering
- photon energy unchanged ("*coherent scattering*")
  good approximation for scattering by non-relativistic e

define **scattering coefficient** $\varsigma_\nu = n_{\rm scat}\sigma_{{\rm scat},\nu}$,
and thus also scattering cross section $\sigma_{\rm scat}$, such that intensity lost to scattering *out* of ray is

$$dI_\nu = -\varsigma_\nu \; I_\nu \; ds \tag{1}$$

isotropic scattering $\rightarrow \varsigma_\nu$ same for all directions

Q: what is intensity scattered into *the ray?*

Isotropic Coherent Scattering

intensity scattered *out* of ray $I_\nu(\hat{n}')$ is

$$dI_\nu(\hat{n}') = -\varsigma_\nu \; I_\nu(\hat{n}') \; ds \tag{2}$$

if scattering *isotropic*, the portion *into* $\hat{n}$ is

$$dI_\nu(\hat{n}) = \frac{d\Omega'}{4\pi} \left| dI_\nu(\hat{n}') \right| \tag{3}$$

and so integrating over all possible solid $d\Omega'$ gives

$$dI_\nu(\hat{n}) = \frac{\varsigma_\nu}{4\pi} \int I_\nu \; d\Omega \; ds = \varsigma_\nu \; J_\nu \; ds \tag{4}$$

where J_ν is the angle-averaged intensity and thus for isotropic coherent scattering

$$\frac{dI_\nu(\hat{n})}{ds} = -\varsigma_\nu \left[I_\nu(\hat{n}) - J_\nu \right] \tag{5}$$

and so the source function is

$$S_\nu = J_\nu \tag{6}$$

and the transfer equation can be written

$$\frac{dI_\nu(\hat{n})}{d\tau_\nu} = -I_\nu(\hat{n}) + J_\nu \tag{7}$$

where mean flux $J_\nu = \int I_\nu(\hat{n}')d\Omega'/4\pi$, and $d\tau_\nu = \varsigma_\nu \; ds$

Q: why is this intuitively correct?

Q: what is effect on I_ν of many scattering events?

for coherent, isotropic scattering:

$$\frac{dI_\nu(\hat{n})}{d\tau_\nu} = -I_\nu(\hat{n}) + J_\nu \tag{8}$$

depends on I_ν field in *all directions*

$\Rightarrow$ scattering couples intensity in different directions

if many scattering events, τ_ν large: $I_\nu \to J_\nu$

after large number of mean free paths, photons $\to$ isotropic

$\Rightarrow$ (isotropic) scattering randomizes photon directions
reduces anisotropy

transfer with scattering: integro-differential equation
generally very hard to solve!

Q: transfer equation modification for anisotropic scattering?

Scattering and Random Walks

Can we understand photon propagation with isotropic scattering in a simple physical picture?

simple model: **random walk**
between collisions, photons move in straight "steps"
with random displacement $\widehat{\ell}$
position after N collisions ("steps") is $\vec{r}_N$

idealizations:

- *step length uniform*: $|\widehat{\ell}| = \ell_{\rm mfp}$ mean free path
- *step direction random*: each $\widehat{\ell}$ drawn from isotropic distribution and independent of previous steps

∞

- initial condition: start at center, $\vec{r}_0 = 0$

1. first step $\vec{r}_1 = \hat{\ell}$

length $|\vec{r}_1| = \ell_{\rm mfp}$, direction random

average over ensemble of photons:

- $\langle \vec{r}_1 \rangle = 0$
- $\langle r_1^2 \rangle = \ell_{\rm mfp}^2$

average positions for *ensemble* of photons is zero

but average distance of *each* photon $\ell_{\rm mfp}$

2. step N has: $\vec{r}_N = \vec{r}_{N-1} + \hat{\ell}$

average over ensemble of photons:

$\langle \vec{r}_N \rangle = \langle \vec{r}_{N-1} \rangle + \langle \hat{\ell} \rangle = \langle \vec{r}_{N-1} \rangle$

but by recursion

$$\langle \vec{r}_N \rangle = \langle \vec{r}_{N-1} \rangle = \langle \vec{r}_{N-2} \rangle = \ldots = \langle \vec{r}_1 \rangle = 0 \tag{9}$$

$\longrightarrow$ ensemble average of photons displacements still 0

as it must be by symmetry

but what about *mean square* displacement?

$$r_N^2 = \vec{r}_N \cdot \vec{r}_N \tag{10}$$
$$= r_{N-1}^2 + 2\hat{\ell} \cdot \vec{r}_{N-1} + \ell_{\rm mfp}^2 \tag{11}$$

average over photon ensemble

$$\langle r_N^2 \rangle = \langle r_{N-1}^2 \rangle + 2\langle \hat{\ell} \cdot \vec{r}_{N-1} \rangle + \ell_{\rm mfp}^2 \tag{12}$$

Q: what is $\langle \hat{\ell} \cdot \vec{r}_{N-1} \rangle$*?*

$$\langle r_N^2 \rangle = \langle r_{N-1}^2 \rangle + 2\langle \hat{\ell} \cdot \vec{r}_{N-1} \rangle + \ell_{\rm mfp}^2 \qquad (13)$$

each photon scattering direction independent from previous

$\langle \hat{\ell} \cdot \vec{r}_{N-1} \rangle = \ell_{\rm mfp} r_{N-1} \langle \cos\theta \rangle = 0$

so $\langle r_N^2 \rangle = \langle r_{N-1}^2 \rangle + \ell_{\rm mfp}^2$

but this means $\langle r_N^2 \rangle = N \ell_{\rm mfp}^2$

$\rightarrow$ **each** photon goes r.m.s. distance

$$r_{\rm rms} = \sqrt{\langle r_N^2 \rangle} = \sqrt{N}\ \ell_{\rm mfp} \qquad (14)$$

so imagine photons generated at $r = 0$

and, after scattering, are observed at distance L

Q: number N of scatterings if optically thin? thick?

Photon Random Walks and Optical Depth

if travel distance L by random walk
then after N scatterings $L = \sqrt{N}\ \ell_{\rm mfp}$
but photon optical depth is $\tau = L/\ell_{\rm mfp}$
$\rightarrow$ counts number of mean free paths in length L

optically thick: $\tau \gg 1$
many scattering events $\rightarrow$ *this is a random walk!*
$N \stackrel{\rm thick}{\approx} \tau^2$

if *optically thin*: $\tau \ll 1$
scattering probability $1 - e^{-\tau} \approx \tau \ll 1$: *not random walk!*
mean number of scatterings over L is $N \stackrel{\rm thin}{\approx} \tau$

approximate expression good for all τ

$$N \approx \tau + \tau^2 \tag{15}$$

Combined Scattering and Absorption

generally, matter can both scatter and absorb photons
transfer equation must include both
for *coherent isotropic scattering* of *thermal radiation*

$$\frac{dI_\nu}{ds} = -\alpha_\nu(I_\nu - B_\nu) - \varsigma_\nu(I_\nu - J_\nu) \tag{16}$$

giving a source function

$$S_\nu = \frac{\alpha_\nu B_\nu + \varsigma_\nu J_\nu}{\alpha_\nu + \varsigma_\nu} \tag{17}$$

a *weighted average* of the two source functions

thus we can write

$$\frac{dI_\nu}{ds} = -(\alpha_\nu + \varsigma_\nu)(I_\nu - S_\nu) \tag{18}$$

with **extinction coefficient** $\alpha_\nu + \varsigma_\nu$

generalize mean free path:

$$\ell_{\mathrm{mfp},\nu} = \frac{1}{\alpha_\nu + \varsigma_\nu} \tag{19}$$

average distance between photon interactions

in random walk picture:
probability of step ending in *absorption*

$$\epsilon_\nu \equiv \alpha_\nu \ell_{\mathrm{mfp},\nu} = \frac{\alpha_\nu}{\alpha_\nu + \varsigma_\nu} \tag{20}$$

and thus step *scattering probability*

$$\varsigma_\nu \ell_{\mathrm{mfp},\nu} = \frac{\varsigma_\nu}{\alpha_\nu + \varsigma_\nu} = 1 - \epsilon_\nu \tag{21}$$

also known as **single scattering albedo**

source function:

$$S_\nu = \epsilon_\nu B_\nu + (1 - \epsilon_\nu) J_\nu \tag{22}$$

Random Walk with Scattering and Absorption

in *infinite medium*: every photon created is eventually absorbed
typical absorption path $\ell_{\text{abs},\nu} = 1/\alpha_\nu$
typical number of scattering events until absorption is

$$N_{\text{scat}} = \frac{\ell_{\text{abs},\nu}}{\ell_{\text{mfp},\nu}} = \frac{\varsigma_\nu + \alpha_\nu}{\alpha_\nu} = \frac{1}{\epsilon_\nu} \tag{23}$$

so typical distance travelled between creation and absorption

$$\ell_* = \sqrt{N_{\text{scat}}}\ell_{\text{mfp},\nu} = \sqrt{\ell_{\text{abs},\nu}\ell_{\text{mfp},\nu}} = \frac{1}{\sqrt{\alpha_\nu(\alpha_\nu + \varsigma_\nu)}} \tag{24}$$

diffusion/thermalization length or *effective mean free path*

What about a *finite medium* of size s?
define optical thicknesses $\tau_{\text{scat}} = \varsigma_\nu s$, $\tau_{\text{abs}} = \alpha_\nu \, s$
and $\tau_* = s/\ell_* = \tau_{\text{scat}}^{1/2}(\tau_{\text{scat}} + \tau_{\text{abs}})^{1/2}$

Q: expected behavior if $\tau_* \ll 1$*?* $\tau_* \gg 1$*?*

$\tau_* = s/\ell_*$: path in units of photon travel until absorption

$\tau_* \ll 1$: *effectively thin* or *translucent*
photons random walk by scattering, seen before absorption
luminosity of thermal source with volume V is

$$L_\nu \overset{\text{thin}}{=} 4\pi\alpha_\nu B_\nu V = 4\pi j_\nu(T)V \tag{25}$$

$\tau_* \gg 1$: *effectively tick*
thermally emitted photons scattered then absorbed before seen
expect $I_\nu \to S_\nu \to B_\nu$
rough estimate of luminosity of thermal source:
most emission from "last scattering" surface of area A
where photons travel $s = \ell_*$

$$L_\nu \overset{\text{thick}}{\approx} 4\pi\alpha_\nu B_\nu \ell_* A \approx 4\pi\epsilon_\nu B_\nu A \tag{26}$$

Mini-Break: Image of the Day