Astro 501: Radiative Processes
Lecture 9
Feb 4, 2013

Announcements:

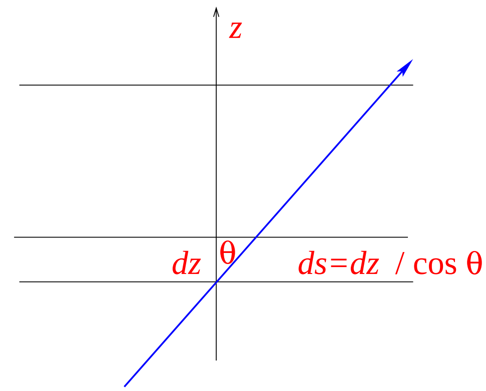
- **Problem Set 3** available, due Friday

Last time: scattering
isotropic coherent scattering *Q: what's that? transfer eq?*
random walk *Q: what's that? rms progress after N steps?*
scattering and absorption: absorption probability, albedo *Q: what's that?*

Today: scattering in a "fluid" approximation
$\rightarrow$ heat flux and the Rosseland mean
begin classical electromagnetic radiation

Radiative Diffusion: Rosseland Approximation

Imagine a **plane-parallel medium**:
n, ρ, T only depend on z
Think: interior of a star



photon propagation depends only on angle θ
between path direction and $\hat{z}$ *Q: why? why not on* ϕ *too?*

change to variable $\mu = \cos\theta$, and note that
path element $ds = dz/\cos\theta = dz/\mu$, so

$$\mu \frac{\partial I_\nu(z,\mu)}{\partial z} = -(\alpha_\nu + \varsigma_\nu)(I_\nu - S_\nu) \tag{1}$$

note: deep inside a real star like the Sun, $\ell_* \sim 1$ cm $\ll R_\star$
Q: implications?

$\ell_* \sim 1$ cm $\ll R_\star$: rapid thermalization, damping of anisotropy

expect stellar interior to have intensity field that
- changes slowly compared to mean free path
- is nearly isotropic

so to *zeroth order* in ℓ_*, transfer equation

$$I_\nu = S_\nu - \mu \ell_* \frac{\partial I_\nu(z,\mu)}{\partial z} \tag{2}$$

gives

$$I_\nu^{(0)} \approx S_\nu^{(0)}(T) \tag{3}$$

this is angle-independent, so: $J_\nu^{(0)} = S_\nu^{(0)}$ and $I_\nu^{(0)} = S_\nu^{(0)} = B_\nu$

Iterate to get *first-order approximation*

$$I_\nu^{(1)} \approx S_\nu^{(0)} - \mu \ell_* \partial_z I_\nu^{(0)} = B_\nu - \frac{\mu}{\alpha_\nu + \varsigma_\nu} \partial_z B_\nu \tag{4}$$

what angular pattern does this intensity field have? why?

to first order, intensity pattern

$$I_\nu^{(1)} \approx S_\nu^{(0)} - \mu \ell_* \partial_z I_\nu^{(0)} = B_\nu - \frac{\mu}{\alpha_\nu + \varsigma_\nu} \partial_z B_\nu \tag{5}$$

i.e., a dominant isotropic component plus
small correction $\propto \mu = \cos\theta$: a *dipole!*
if T decreases with z, then $\partial_z B_\nu < 0$, and so
intensity brighter downwards (looking into hotter region)

use this find **net specific flux along** z

$$F_\nu(z) = \int I_\nu^{(1)}(z,\mu)\ \cos\theta\ d\Omega = 2\pi \int_{-1}^{+1} I_\nu^{(1)}(z,\mu)\ \mu\ d\mu \tag{6}$$

only the *anisotropic* piece of $I_\nu^{(0)}$ of survives *Q: why?*

$$F_\nu(z) = -\frac{2\pi}{\alpha_\nu + \varsigma_\nu} \partial_z B_\nu \int_{-1}^{+1} \mu^2\ d\mu \tag{7}$$

$$= -\frac{4\pi}{3(\alpha_\nu + \varsigma_\nu)} \partial_z B_\nu \tag{8}$$

net specific flux along z

$$F_\nu(z) = -\frac{4\pi}{3(\alpha_\nu + \varsigma_\nu)} \partial_z B_\nu = -\frac{4\pi}{3(\alpha_\nu + \varsigma_\nu)} \partial_T B_\nu \ \partial_z T \tag{9}$$

since $B_\nu = B_\nu(T)$

total integrated flux

$$F(z) = \int F_\nu(z) \ d\nu = -\frac{4\pi}{3} \partial_z T \int (\alpha_\nu + \varsigma_\nu)^{-1} \frac{\partial B_\nu}{\partial T} \ d\nu \tag{10}$$

to make pretty, note that

$$\int \partial_T B_\nu \ d\nu = \partial_T \int B_\nu \ d\nu = \partial_T B(T) = \frac{4\pi\sigma T^3}{\pi} \tag{11}$$

and define **Rosseland mean absorption coefficient**

$$\frac{1}{\alpha_{\mathrm{R}}} = \frac{\int (\alpha_\nu + \varsigma_\nu)^{-1} \partial_T B_\nu \ d\nu}{\int \partial_T B_\nu \ d\nu} \tag{12}$$

average effective mean free path, weighted by Planck derivative

Energy Flux in the Rosseland Approximation

using Rosseland mean, we have

$$F(z) = -\frac{16\sigma T^3}{3\alpha_{\mathsf{R}}}\frac{\partial T}{\partial z} \tag{13}$$

Rosseland approximation to radiative flux

Q: what if T uniform? decreasing upwards? implications for stars?

Note:

- whenever energy (heat) flux $\vec{F} - \chi \nabla T$
  coefficient χ is the *heat conductivity*
- in the presence of a heat flux, thermal energy density changes:

$$\partial_t u = -\nabla \cdot \vec{F} \tag{14}$$

a continuity equation, i.e., local statement of energy conservation
for radiation, $u = u(T)$, so $\partial_t T \sim D \nabla^2 T$: a diffusion equation!

in stars, energy must be transported from interior
where it is created by thermonuclear reactions
upwards until it is radiated to space

in regions when temperature gradient $\partial_z T$ not too large
radiative diffusion is the mechanism for energy transport
i.e., photons random walk their way out of the star

- typical solar photon is millions of years old
- unlike neutrinos which are minutes old

photon *luminosity* in interior radius r is

$$L(r) = 4\pi r^2 \ F(r) = -4\pi r^2 \frac{16\sigma T^3}{3\alpha_{\rm R}} \frac{\partial T}{\partial r} \tag{15}$$

solar temperature drops with radius, $\partial_z T < 0$,
so $L > 0$: energy flows outwards!

Classical Electromagnetic Radiation

Electromagnetic Forces on Particles

Consider *non-relativistic classical particle*
with mass m, charge q and velocity $\vec{v}$

under an electric field $\vec{E}$ and magnetic field $\vec{B}$
the particle feels a **force**

$$\vec{F} = q\ \vec{E} + q\ \frac{\vec{v}}{c} \times \vec{B} \tag{16}$$

sums Coulomb and Lorentz forces
units: cgs throughout; has nice property that $[E] = [B]$

power supplied by EM fields to charge

$$\frac{dU_{\text{mech}}}{dt} = \vec{v} \cdot \vec{F} = q\ \vec{v} \cdot \vec{E} = \frac{d}{dt}\frac{mv^2}{2} \tag{17}$$

no contribution from $\vec{B}$: "magnetic fields do no work"

Q: what if smoothly distributed charge density and velocity field?

Electromagnetic Forces on Continuous Media

consider a medium with charge density ρ_q
and current density $\vec{j} = \rho_q \vec{v}$

by considering an "element" of charge $dq = \rho_q \; dV$
we find **force density**, defined via $d\vec{F} = \vec{f} \; dV$:

$$\vec{f} = \rho_q \; \vec{E} + \frac{\vec{j}}{c} \times \vec{B} \tag{18}$$

and a **power density** supplied by the fields

$$\frac{\partial u_{\mathrm{mech}}}{\partial t} = \vec{j} \cdot \vec{E} \tag{19}$$

note: if medium is a collection of point sources $q_i, \vec{r}_i, \vec{v}_i$

$$\rho_q(\vec{r}) = \sum_i q_i \; \delta(\vec{r} - \vec{r}_i) \tag{20}$$

and current density is

$$\vec{j}(\vec{r}) = \sum_i q_i \; \vec{v}_i \; \delta(\vec{r} - \vec{r}_i) \tag{21}$$

Maxwell's Equations

Maxwell relates fields to charge and current distributions

in the absence of dielectric media ($\epsilon = 1$)
or permeable media ($\mu = 1$):

$$\begin{array}{rcll} \nabla \cdot \vec{E} & = & 4\pi\rho_q & \text{Coulomb's law} \\ \nabla \cdot \vec{B} & = & 0 & \text{no magnetic monopoles} \\ \nabla \times \vec{E} & = & -\frac{1}{c}\partial_t \vec{B} & \text{Faraday's law} \\ \nabla \times \vec{B} & = & \frac{4\pi}{c}\vec{j} + \frac{1}{c}\partial_t \vec{E} & \text{Ampère's law} \end{array} \tag{22}$$

take divergence of Ampère

$$\partial_t \rho_q + \nabla \cdot \vec{j} = 0 \tag{23}$$

conservation of charge!

now can rewrite power exerted by fields on charges
in terms of fields only *Q: how?*

Field Energy

Power density exerted by fields on charges

$$\frac{\partial u_{\text{mech}}}{\partial t} = \vec{j} \cdot \vec{E} = \frac{1}{4\pi} \left(c \nabla \times \vec{B} - \partial_t \vec{E} \right) \cdot \vec{E} \tag{24}$$

with clever repeated use of Maxwell,
can recast in this form:

$$\frac{\partial u_{\text{fields}}}{\partial t} + \nabla \cdot \vec{S} = -\frac{\partial u_{\text{mech}}}{\partial t} \tag{25}$$

Q: physical significance of eq. (25)?

energy change per unit time

$$\frac{\partial u_{\text{fields}}}{\partial t} + \nabla \cdot \vec{S} = -\frac{\partial u_{\text{mech}}}{\partial t} \tag{26}$$

reminiscent of $\partial_t \rho_q + \nabla \cdot \vec{j} = 0$
$\rightarrow$ an expression of local conservation of energy
where the mechanical energy acts as source/sink

identify **electromagnetic field energy density**

$$u_{\text{fields}} = \frac{E^2 + B^2}{8\pi} \tag{27}$$

i.e., $u_E = E^2/8\pi$, and $u_B = B^2/8\pi$

and **Poynting vector** is *flux of EM energy*

$$\vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{B} \tag{28}$$

this is huge for us ASTR 501 folk! EM flux!
Q: when zero? nonzero? direction?

Electromagnetic Waves

in vacuum ($\rho_q = 0 = \vec{j}$), and in Cartesian coordinates Maxwell's equations imply (PS3):

$$\nabla^2 \vec{E} - \frac{1}{c^2}\partial_t^2 \vec{E} = 0 \quad (29)$$

$$\nabla^2 \vec{B} - \frac{1}{c^2}\partial_t^2 \vec{B} = 0 \quad (30)$$

both fields satisfy a **wave equation**

wave equation invites **Fourier transform** of fields:

$$\vec{E}(\vec{k},\omega) = \frac{1}{(2\pi)^2}\int d^3\vec{r}\ dt\ \ \vec{E}(\vec{x},t)\ e^{-i(\vec{k}\cdot\vec{r}-\omega t)} \quad (31)$$

inverse transformation:

$$\vec{E}(\vec{x},t) = \frac{1}{(2\pi)^2}\int d^3\vec{k}\ d\omega\ \ \vec{E}(\vec{k},\omega)\ e^{i(\vec{k}\cdot\vec{r}-\omega t)} \quad (32)$$

note symmetry between transformation (but sign flip in phase!)

original real-space field can be expressed as

$$\vec{E}(\vec{x},t) = \frac{1}{(2\pi)^2} \int d^3\vec{k}\ d\omega \quad \vec{E}(\vec{k},\omega)\ e^{i(\vec{k}\cdot\vec{r}-\omega t)} \tag{33}$$

expansion in *sum of Fourier modes* with

- **wavevector** $\vec{k}$
  magnitude $k = 2\pi/\lambda$, direction $\hat{n} = \vec{k}/k$
- **angular frequency** $\omega = 2\pi\ \nu$

apply wave equation to Fourier expansion:

$$\nabla^2\vec{E} - \frac{1}{c^2}\partial_t^2\vec{E} = -\frac{1}{(2\pi)^2 c^2}\int d^3\vec{k}\ d\omega\ (c^2k^2 - \omega^2)\ \vec{E}(\vec{k},\omega)\ e^{i(\vec{k}\cdot\vec{r}-\omega t)} \tag{34}$$

$$= 0 \tag{35}$$

for notrivial solutions with $\vec{E} \neq 0$,
this requires $\omega^2 = c^2k^2$, or **vacuum dispersion relation**

$$\omega = ck \tag{36}$$

i.e., wave solutions require constant phase velocity $v_\phi = \omega/k = c$