Astronomy 507 Spring 2014
Problem Set #4

Due in class: Friday, March 21; or on Compass Monday March 24
Total points: 10+1

Throughout this problem set, whenever a cosmological parameters need to be chosen, we will adopt the concordance cosmology, i.e., a flat $\Omega_{\rm m,0} = 0.3$, $\Omega_{\Lambda,0} = 0.7$ universe with $\eta = 6 \times 10^{-10}$.

1. *Cosmic Thermal Photodissociation.*

   (a) **[1 bonus point]** In several cosmic situations we will want to know the number density of thermal photons (or other relativistic particles) with energies exceeding some scale ϵ which lies above the particles' temperature T. That is, we are interested in the number density of photons in the "high-energy tail." Show that, for $\epsilon \gg T$, the number density of particles with energies above ϵ is

   $$n_{\rm rel}(> \epsilon) \approx \frac{g}{2\pi^2} \epsilon^2 T e^{-\epsilon/T} \quad (1)$$

   Hint: in the integral over phase space, it may be useful to change variables from p to $q = p - \epsilon$.

   Finally, eq. (1) is written in units where $\hbar = c = k_{\rm Boltz} = 1$. Revise eq. (1), replacing these factors as needed to restore the correct observable units for n.

   (b) **[1 point]** We typically will want to use eq. (1) to compute the number of high-energy, dissociating photons per baryon. In particular, it is of interest to find when $n_{\rm rel}(> \epsilon)$ drops below the baryon number density $n_{\rm b}$. Find an expression which shows how the temperature at which this occurs, $T_{\rm dis}$, is related to the energy scale ϵ and the baryon-to-photon ratio η.

   Give an approximate expression for $T_{\rm dis}$, ignoring logarithmic corrections (i.e., ignore terms like $\ln T$).

   Finally, consider the case in which ϵ is the energy needed for photons to break up, or "dissociate," a bound state of baryons. Interpret the physical significance $T_{\rm dis}$ and explain why it is not just ϵ.

2. *The Epoch of Recombination.*

   (a) **[1 point]** Estimate the temperature of recombination, using your result from question 1b above. Be sure to explain your choice of energy scale ϵ. Go on to estimate the redshift $z_{\rm rec}$ of recombination.

   (b) **[1 point]** To make a more refined estimate, calculate the redshift $z_{\rm rec}$ of recombination, using the Saha equation. Take as the condition for recombination that only a fraction $X_{e,\rm rec} = 10\%$ of the electrons remain free.

   Compare your result to the estimate from part 3(a), and comment.

   (c) **[1 point]** Show how your result from part (b) would change if you had defined recombination by free electron fractions of $X_{e,\rm rec} = 50\%$, or $X_{e,\rm rec} = 1\%$.

Comment on the result and give your estimate of the quantitative uncertainty or "fuzziness" $\Delta z_{\rm rec}$ in the recombination redshift due to this arbitrariness in assigning a unique instant to this continuous (but rapid) event.

3. *Angular Diameter of the Recombination Horizon.*

   (a) **[1 point]** Find an expression for the angular diameter distance to an object whose redshift z falls within the matter-dominated epoch of a concordance cosmology. Evaluate the result numerically for $z = z_{\rm rec}$.

   Then find an expression for the physical (i.e., not comoving) size of the particle horizon for an instant whose redshift z falls within the matter-dominated epoch of a concordance cosmology. Evaluate the result numerically for $z = z_{\rm rec}$.

   (b) **[1 point]** Given the results for parts (a) and (b), calculate an expression for the angular diameter $\theta_{\rm hor,rec}$ of the horizon at recombination in terms of $z_{\rm rec}$. Evaluate your result numerically and express it in degrees.

   Finally, interpret your result physically: on the basis of your calculations (i.e., don't worry yet about inflation) how would you understand CMB temperature differences (or sameness) observed on angular scales smaller than $\theta_{\rm hor,rec}$? larger than $\theta_{\rm hor,rec}$?

4. *BBN: Running the Numbers.*

   (a) **[1 point]** In class we argued that essentially all cosmic neutrons which survive immediately *prior* to helium formation during BBN ultimately go into ^{4}He nuclei. Show that if this is the case, then the resulting helium mass fraction $Y_p = \rho(^4{\rm He})/\rho({\rm baryon})$ is related to the prior n/p ratio via

$$Y_p = 2\frac{(n/p)}{1+(n/p)} \tag{2}$$

   You may take $m(^4{\rm He}) = 4m_p$.

   Find Y_p for $n/p \approx 1/7$, and comment on the result.

   (b) **[1 point]** Today we know that the photons (the CMB) have a number density $n_{\gamma,0} \propto T_{\gamma,0}^3 = 411\ {\rm photons/cm^3}$. Given this, and a value for η, find an expression the present baryon mass density $\rho_{B,0}$. You may assume for simplicity they all baryons are proton, i.e., each has a mass $m_{\rm baryon} = m_p$.

   Go on to find an expression for $\Omega_B h^2$ in terms of η, $n_{\gamma,0}$, and other known constants/parameters. Evaluate $\Omega_B h^2$ for $\eta = 6.047 \times 10^{-10}$, the recently-published *Planck* result.

5. *Big Bang Nucleosynthesis: Deuterium Production.* The first complex nucleus made in the universe is deuterium d, created by the radiative capture of a neutron on a proton: $n + p \to d + \gamma$. The energy liberated in the reaction is $Q = 2.22$ MeV. The spin of the deuteron is $S_d = 1$.

   This reaction is the first step in building the light elements, and thus it is the "nuclear gateway" to big bang nucleosynthesis. We wish to understand how the cosmic deuteron abundance grows, due to this reaction.

(a) **[1 point]** Deuterium production is at first stymied by the reverse reaction, i.e., photodissociation $d + \gamma \to p + n$. Estimate the temperature T_d at which deuteron dissociation ceases, using your result from question 1(b) above. Be sure to explain your choice of energy scale ϵ.

(b) **[1 point]** To improve this estimate, now use the Saha equation to find the equilibrium "baryon fraction" $Y_d = n_d/n_\mathrm{b}$ of deuterium, in terms of the baron fractions Y_n, Y_p of neutrons and protons, η, and physical constants. Use this to show that, at high temperatures, Y_d is vanishingly small. You may take Y_p and Y_n to be of order unity.

Go on to show that Y_d grows as temperature decreases. Estimate the temperature T_d at which Y_d would become large (i.e., of order unity). This epoch marks the onset of the synthesis of the light elements.