Astronomy 507 Spring 2014
Problem Set #5

Due in class: **Friday**, April 11

Total points: 10+2

1. *Hot Relics: Neutrinos.* Assume that the known neutrino species (e, μ, τ) have masses such that $m \ll 1$ MeV, but $m \gg T_0$.

   (a) **[1 point]** In class we considered cold relics, for which $T_f \ll m$. Neutrinos, however, are hot relics. Consider a fermionic species ψ that is relativistic, so that $T_f \gg m$, that has $\mu_\psi = 0$, and that has a number g_ψ of internal degrees of freedom. Find the thermodynamic equilibrium (thermal) abundance $Y_{\rm eq} = n_\psi / n_\gamma$.

   Comment on how $Y_{\rm eq}$ depends (or does not depend!) on the freezeout temperature (always assuming the species remains relativistic at freezeout).

   (b) **[1 point]** For all species of neutrinos, the annihilation cross section is of the same order as the $n \leftrightarrow p$ cross section mentioned in class: $\sigma_{\rm ann} \simeq \sigma_0 (E/m_e)^2$, where $\sigma_0 \simeq 2 \times 10^{-44}$ cm^2. Find an expression for $\langle \sigma v \rangle_{\rm ann}$ as a function of T.

   (c) **[1 point]** Using the result from (b), calculate the neutrino freezeout temperature T_f. If each species i has mass m_i, find its present relic abundance $Y_i = n_{\nu_i}/n_\gamma$ (assuming it is non-relativistic today, $m_\nu \gg T_0$).

   (d) **[1 bonus point]** Show that today, $T_\nu / T_\gamma = (4/11)^{1/3}$. To do this, assume that neutrino freezeout occurs entirely before any $e^\pm$ annihilation. This means that there is no energy (heat) exchange between neutrinos and the electromagnetic plasma during $e^\pm$ pair annihilation, and thus comoving neutrino entropy $S_\nu = s_\nu a^3$ stays the same before and after. Show that this implies $T_\nu \propto 1/a$.

   Then consider the comoving entropy $S_{\rm EM} = a^3 s_{\rm EM}$ in relativistic electromagnetic particles. Well before annihilation, these are γ and $e^\pm$ pairs, and after only γ. Equate $S_{\rm EM}$ well before annihilation and well after annihilation, with photon and neutrino temperatures $(T_i, T_{\nu,i})$ and $(T_f, T_{\nu,f})$. Show that the neutrino temperature is related to photon temperature by

   $$\frac{T_{\nu,f}}{T_{\nu,i}} = \left(\frac{g_{*,S,f}^{\rm EM}}{g_{*,S,i}^{\rm EM}} \right)^{1/3} \frac{T_f}{T_i} = \left(\frac{4}{11} \right)^{1/3} \frac{T_f}{T_i} \tag{1}$$

   Finally, note that before neutrino decoupling, $T_\nu = T$. Show trivially that this means that after pair annihilation, $T_\nu = (4/11)^{1/3} T$.

   (e) **[1 point]** Use the result from (c) to show that the (very generous) condition $\Omega_\nu \leq 1$ corresponds to a limit on neutrino mass, which you should find to be about

   $$\sum_{\rm neutrinos} m_i \lesssim 50 \text{ eV} \tag{2}$$

   Note: to get the right value you have to use the fact that today, $T_\nu / T_\gamma = (4/11)^{1/3}$.

   How does this compare to Particle Data Group constraints on neutrino masses? (See links from course page, and go to the "Neutrino Properties entry under the Leptons heading of "Particle Listings.")

2. *Observational Requirements for Inflation: Cosmic e-foldings.*

   (a) **[1 point]** Imagine that we have information which tells us that the universe pass through an cosmic epoch in either the radiation or matter eras. It follows that the universe didn't recollapse or go to zero density soon thereafter, and that we have nearly flat universe today, so that the curvature then must have been small. Given some epoch z, and the current limits $\|\Omega_{\kappa,0}\| \equiv \|\Omega_0 - 1\| \leq 0.01$, find and expression for the limits on the curvature parameter $\|\Omega_\kappa\| \equiv \|\Omega(z) - 1\|$. Note that the results are different depending on whether the epoch is matter- or radiation-dominated.

   One way to state the flatness problem is that "generically" one expects the curvature term comparable to the others: $\|\Omega - 1\| \sim 1$, while you have found $\|\Omega - 1\| \ll 1$. Use your result to deduce the required number N_{min} of inflationary e-foldings prior to the epoch z in order to leave it as flat as you have required. To do this, assume that prior to inflation, the generic condition $\|\Omega - 1\| \sim 1$ held. Then you can calculate how much the curvature would need to be inflated to meet some observed bound on $\|\Omega - 1\|$.

   (b) **[1 point]** Apply your result from (a) to find N_{min} as implied by these cosmic epochs: recombination, BBN, the "Fermilab era" when $T \sim E_{\text{Tevatron}} = 1$ TeV, the GUT era $\sim 10^{15}$ GeV, and the Planck epoch.

   (c) **[1 bonus point]** (Following Liddle & Lyth 3.5) If the universe underwent a GUT transition $T \sim 10^{15}$ GeV), it is expected that one magnetic monopole ($m \sim 10^{15}$ GeV) was created per Hubble volume. In the absence of inflation, compute the relic mass density of monopoles today; you should get an uncomfortably large number. Using the limit $\Omega_{\text{Monopole},0} \lesssim 10^{-6}$ (Parker bound), compute require the number of e-foldings of inflation needed to respect this bound. Compare your result to those above, and comment.

3. *Scalar Field Dynamics.* A classical and spatially homogeneous scalar field ϕ which only interacts with itself (via a potential V) and with gravity has an equation of motion in a FRW universe given by

$$\ddot{\phi} + 3H\dot{\phi} + dV/d\phi = 0 \tag{3}$$

   (a) **[1 point]** In a non-expanding universe, show that the equation of motion implies that $\rho_\phi = \dot{\phi}^2/2 + V(\phi)$ is a constant.

   Find an expression for $\dot{\rho}_\phi$ in an expanding universe, and interpret it physically.

   (b) **[1 point]** Show that if the kinetic term dominates the ϕ energy density (i.e., if V is negligible), $\rho_\phi \propto a^{-6}$. Also find the value of w_ϕ in this case.

   If the kinetic term in ρ_ϕ not only dominates V but also the rest of the energy density in the universe ("kination"), go on to find the time evolution $\phi(t)$.

   It is not known if the universe ever underwent such a phase. Comment on why such a phase is unsuitable for inflation.

4. *Slow-Roll Conditions.*

   (a) **[1 point]** Show that the slow-roll requirements that $\dot{\phi}^2/2 \ll V(\phi)$ and $\ddot{\phi} \ll 3H\dot{\phi}$ are equivalent to the statements that

$$\epsilon(\phi) \equiv \frac{m_{\rm Pl}^2}{2}\left(\frac{V'}{V}\right)^2 \ll 1 \qquad (4)$$

$$\|\,\eta(\phi)\,\| \equiv m_{\rm Pl}^2 \left\|\frac{V''}{V}\right\| \ll 1 \qquad (5)$$

   where η here is neither the baryon-to-photon ratio nor conformal time! Also, here and throughout, we follow Liddle & Lyth in using the "reduced Planck mass" $m_{\rm Pl} = M_{\rm Pl}/\sqrt{8\pi} = \sqrt{\hbar c/8\pi G}$, so that, e.g., Friedmann reads $H^2 = \rho/3m_{\rm Pl}^2$.

   (b) **[1 point]** Show that if both ϵ and η are strict constants, independent of ϕ, this uniquely specifies the inflation potential to be of the form

$$V(\phi) = V_0 e^{\phi/\mu} \qquad (6)$$

   Find the value of the energy scale μ in terms of ϵ, η, and physical constants.