Astronomy 507 Spring 2014
Problem Set #6: Structure Formation

Due in class: Wednesday, April 30

Total points: 10+3

1. **[1 bonus point]** *Density Contrasts: Numerical Values.* Density contrasts are of course dimensionless, but it is still a good idea to get a feel for the numbers. Estimate the present-day density contrast δ of:

   - a typical galaxy cluster
   - our location in the Milky Way
   - the interstellar medium of the Milky Way
   - the best vacuum that can be created in a terrestrial laboratory
   - yourself

   Comment on these results.

2. *Linear Perturbations: Baryon Oscillations and the CMB.*

   (a) **[1 point]** Consider the universe in the matter-dominated era, and focus on the dark matter and baryon fluids. Show that in the linear regime, the baryonic density contrast $\delta_{\rm b}$ of comoving wavenumber k obeys the evolution equation

   $$\ddot{\delta}_b + 2\frac{\dot{a}}{a}\dot{\delta}_b + \frac{c_s^2 k^2}{a^2}\delta_{\rm b} = 4\pi G\rho(\Omega_m\delta_m + \Omega_b\delta_{\rm b}) \approx 4\pi G\rho\Omega_m\delta_m \tag{1}$$

   where it is assumed that $\Omega_{\rm tot} = 1$. This is a fairly trivial setup for the rest of the problem.

   Go on to consider modes with large wavenumbers $kc_s/a \gg 4\pi G\rho\Omega_m$. Show that this corresponds to scales below the comoving Hubble length. Write the evolution equation for $\delta_{\rm b}$ for such modes.

   (b) **[1 point]** For such large wavenumber modes, argue that the solutions should have an oscillatory character. To see what is going on, let us simplify matters and consider the case of a sound speed c_s which is constant in time.

   Anticipating an oscillatory solution, write $\delta_{\rm b}(t) = A(t)\, e^{i\theta(t)}$, with A and θ both real. Plug this form into the evolution equation you found in (b). The result will be complex and thus really is a set of two equations, one real and one complex.

   For a first approximation, assume $A(t)$ is slowly varying compared to θ (i.e., the solution is mostly just an oscillation). So take A to be a constant, and then the real equation only contains first derivatives, and amounts to a first approximation to the solution. Show that the real part of the equation is satisfied if

   $$\theta(t) = \int kc_s\frac{dt}{a} = kc_s\eta = kd_{\rm hor,s} \tag{2}$$

   where η is the conformal time.

Then refine your solution by keeping $\theta(t)$ as you just found, but now let A vary with time. In this case, show that the imaginary equation gives $A \propto 1/a\sqrt{\dot{\theta}} = 1/\sqrt{ac_s k}$. This means that the solution now becomes $\delta_{\mathrm{b}}(t) = D/\sqrt{ac_s k}\; e^{i\theta(t)}$, with D a constant.

This type of solution is the first step in the WKB approximation; more refined approximations can be made extremely accurate.

Explain why $d_{\mathrm{hor,s}} = c_{\mathrm{s}}\eta$ is known as the "sound horizon." How do $d_{\mathrm{hor,s}}$ and θ scale with cosmic time t, always assuming a matter-dominated universe?

(c) **[1 point]** Examine the physical nature of the solution $\delta_{\mathrm{b}} \sim e^{i\theta}$ as a function of k and of t. For a fixed length scale (fixed k), when is the first compression? The first rarefaction? At a fixed epoch t (or η), what sets the scale which has just compressed for the first time? What is the connection between this scale and those which are at other extrema of compression or rarefaction?

(d) **[1 point]** Prior to recombination, radiation pressure dominates the pressure the baryons feel, and thus the sound speed. By using the different radiation and matter evolution with a, show that a fluid with radiation and pressureless matter has

$$c_s^2 \equiv \frac{dP}{d\rho} = \frac{1}{3}\left(\frac{3}{4}\frac{\rho_m}{\rho_r} + 1\right)^{-1} \tag{3}$$

(as always in units where the speed of light is $c = 1$). What is c_s^2 deeply in the radiation-dominated phase? at matter-radiation equality? At decoupling?

(e) **[1 point]** Finally, combine the last two parts to arrive at the comoving length scale of the largest perturbations in the CMB. How does this compare to the comoving horizon size at recombination?

In light of your results, how can you understand the overall behavior of CMB fluctuations as a function of angular scale?

3. *Nonlinear Perturbations: The Spherical Collapse Model.* Although it is in general impossible to solve analytically for the full nonlinear evolution, for the idealized special case of a spherically symmetric perturbation a full solution is possible. The trick is that a spherically symmetric perturbation with uniform density evolves according to the same equations as a closed (or open) Friedmann universe–one may legitimately think of such a region as an independent "subuniverse."

(a) **[1 bonus point]** Consider a uniform matter-dominated *over*density. For such a region, the radius $r(t) = a(t)R$ evolves with $a(t)$ a solution to the Friedmann equation for a *closed* universe. An analytic solution for $a(t)$ is now available, but there is a parametric solution, in which a and t are related by an auxiliary quantity, the "development angle" θ. For a bonus point, start with the Friedmann equation and derive the solution

$$a(\theta) = A(1 - \cos\theta) \tag{4}$$
$$t(\theta) = B(\theta - \sin\theta) \tag{5}$$

(b) **[1 point]** Interpret the results from part (a) physically. Describe the evolution of an overdensity in the expanding universe.

What is the value of θ, a, and t at maximum expansion (also known as "turnaround")? At final collapse? Write A in terms of $a_{\rm max}$ and B in terms of $t_{\rm coll}$.

(c) **[1 bonus point]** Show that for small t, expanding both a and t to next-to-leading order gives

$$a \simeq \frac{A}{2}\left(\frac{6t}{B}\right)^{2/3}\left[1-\frac{1}{20}\left(\frac{6t}{B}\right)^{2/3}\right] \tag{6}$$

(d) **[1 point]** Now we compare the overdensity evolution to that of a flat, matter-dominated universe (Einstein-de Sitter). In particular, consider such a "background" universe which initially has (essentially) the same density as the perturbed region we have introduced, and thus initially the same expansion rate. Call the scale factor in this background universe $a_{\rm bg}$. Show that, compared to this reference universe, the overdensity has a density contrast

$$\delta(t) = \left(\frac{a_{\rm bg}}{a}\right)^3 - 1 \tag{7}$$

and explain why $\delta(t) \geq 0$ for all times (until complete collapse of the overdensity!).

(e) **[1 point]** Use the results from parts (b), (c), and (d) to show that for small t,

$$\delta(t) \approx \delta_{\rm lin}(t) = \frac{3}{20}\left(\frac{12\pi t}{t_{\rm coll}}\right)^{2/3} \tag{8}$$

This is the first-order approximation to the density contrast. Compare this to our in-class solution to the linearized evolution equation for δ, and comment.

(f) **[1 point]** A major payoff of this exercise is that we now have a way to relate the behavior of the *linearized* density contrast to the full nonlinear contrast for any time $t \leq t_{\rm coll}$. To see how this goes, first find the true value for $1+\delta(t_{\rm max})$ at maximum expansion. Compare this result to that of the linearized density contrast at the same time, $1+\delta_{\rm lin}(t_{\rm max})$.

(g) **[1 point]** After turnaround, the spherical model formally gives a collapse to an infinite density at $t_{\rm coll}$. In practice, what really happens is that the structure would virialize and then maintain a constant radius consistent with virial equilibrium. Show that the virial theorem gives that $a_{\rm virial} = a_{\rm max}/2$. Using this as the overdensity scale factor at collapse, find the true value of $\delta(t_{\rm coll})$ at collapse. Compare this to the linearized value for $\delta(t_{\rm coll})$ at collapse.

Based on your result, comment on why people often define a collapsed object in the universe today to be a region which encloses an overdensity $\delta \sim 180 - 200$ (the criterion differs slightly from author to author).

Also based on your result, comment on the significance of perturbations which today have $\delta_{\rm lin} \geq \delta_c = 1.69$.