Astro 507

Lecture 21

March 9, 2014

Announcements:

- **Preflight 4 due 9am Friday**

Last time: theory of isotropic CMB spectrum

key aspect: Thompson scattering is *only* process acting
for most photons (i.e., for all photons with $h\nu \lesssim 40kT$)

Given a photon spectrum I_ν *prior* to decoupling

Q: what is spectrum after Thompson freezeout?

Observed (post-decoupling) CMB spectrum: *thermal*

Q: implications?

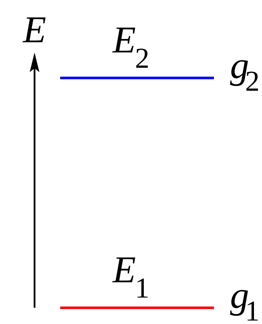
Q: what physically controls onset of decoupling?

Statistical Mechanics and Cosmology

For much of cosmic time contents of U. in *thermal equilibrium*

statistical mechanics: at fixed $T \rightarrow$ matter & radiation n, ρ, P
then cosmic $T(a)$ evolution $\rightarrow n, \rho, P$ at any epcoh

Boltzmann: consider a particle (elementary or composite)
with a series of energy states:
for two sets of states with energies E_1 and $E_2 > E_1$
and degeneracies (# states at each E) g_1 and g_2
ratio of number of particles in these states is



$$\frac{n(E_2)}{n(E_1)} = \frac{g_2}{g_1} e^{-(E_2 - E_1)/T}$$

where I put $k = 1$, i.e., $kT \rightarrow T$

Example: atomic hydrogen, at T
Q: ratio of ground (1S) to 1st excited state (2P) populations?

Atomic hydrogen (H I):

- energy levels: $E_n = -B_{\mathsf{H}}/n^2$ for $n \geq 1$
- angular momenta degeneracies: $g_\ell = 2\ell + 1$

$1S$: $n = 1 \to E(1S) = -B$; $\ell = 0 \to g(1S) = 1$

$2P$: $n = 2 \to E(2P) = -B/4$; $\ell = 1 \to g(2P) = 3$

$$\frac{n(2P)}{n(1S)} = 3e^{-3B/4T} = 3e^{-120{,}000\,\mathsf{K}/T} \tag{1}$$

Q: sanity checks–is this physically reasonable?

Q: how does this ratio change if plasma is partially ionized i.e., contains both H I *and* H II$=$ H$^+ = p$*?*

Note: H is bound system $\to$ discrete energies
we now broaden analysis to include unbound systems
$\to$ continuous energies, momenta

Statistical Mechanics in a Nutshell

classically, **phase space** $(\vec{x}, \vec{p})$
  completely describes particle state
but quantum mechanics $\rightarrow$ uncertainty $\Delta x \Delta p \geq \hbar/2$
semi-classically: min phase space "volume"
$(dx\ dp_x)(dy\ dp_y)(dz\ dp_z) = h^3 = (2\pi\hbar)^3$
per quantum state of fixed $\vec{p}$

define "**occupation number**" or "**distribution function**" $f(\vec{x}, \vec{p})$:
number of particles in each phase space "cell"
Q: f range for fermions? bosons?

$$dN = g f(\vec{x}, \vec{p})\ \frac{d^3\vec{x}\ d^3\vec{p}}{(2\pi\hbar)^3} \tag{2}$$

where g is # internal (spin/helicity) states:
Q: $g(e^-)$? $g(\gamma)$? $g(p)$?

Fermions: $0 \leq f \leq 1$ (Pauli)
Bosons: $f \geq 0$ $g(e^-) = 2s(e^-) + 1 = 2$ electron, same for p
$g(\gamma) = 2$ (polarizations) photon

Particle phase space occupation f determines bulk properties

Number density

$$n(\vec{x}) = \frac{d^3N}{d^3x} = \frac{g}{(2\pi\hbar)^3} \int d^3\vec{p}\ f(\vec{p}, \vec{x}) \tag{3}$$

Mass-energy density

$$\varepsilon(\vec{x}) = \rho(\vec{x})c^2 = \langle En \rangle = \frac{g}{(2\pi\hbar)^3} \int d^3\vec{p}\ E(p)\ f(\vec{p}, \vec{x}) \tag{4}$$

Pressure see director's cut extras for more

$$P(\vec{x}) = \langle p_i v_i n \rangle_{\mathrm{direction}\,i} \stackrel{\mathrm{isotrop}}{=} \frac{\langle pvn \rangle}{3} = \frac{g}{(2\pi\hbar)^3} \int d^3\vec{p}\ \frac{p\,v(p)}{3}\ f(\vec{p}, \vec{x}) \tag{5}$$

Q: these expressions are general–simplifications in FLRW?

FRLW universe:

- homogeneous $\rightarrow$ no $\vec{x}$ dep
- isotropic $\rightarrow$ only $\vec{p}$ magnitude important $\rightarrow$ $f(\vec{p}) = f(p)$

in **thermal equilibrium** at T:

▷ *Boson occupation number*

$$f_{\mathrm{b}}(p) = \frac{1}{e^{(E-\mu)/kT} - 1} \tag{6}$$

▷ *Fermion occupation number*

$$f_{\mathrm{f}}(p) = \frac{1}{e^{(E-\mu)/kT} + 1} \tag{7}$$

Note: μ is "*chemical potential*" or "Fermi energy"
$\mu = \mu(T)$ but is *independent* of E

If $E - \mu \gg T$: both $f_{\mathrm{f,b}} \longrightarrow f_{\mathrm{Boltz}} = e^{-(E-\mu)/kT}$
$\rightarrow$ *Boltzmann distribution*

The Meaning of the Chemical Potential

For a particle species in thermal equilibrium

$$f(p;T,\mu) = \frac{1}{e^{[E(p)-\mu]/kT} \pm 1} \tag{8}$$

What is μ, and what does it mean physically?

First, consider what if $\mu = 0$?

- then f depends only on T and particle mass and thus so do n, ρ, P *Q: why?*
- all samples of a substance at fixed T have exactly the same n, ρ, P!
- and hotter $\rightarrow$ larger n, ρ, P

sometimes true! *Q: examples?*
but not always! *Q: examples?*

Q: what is physics behind μ?

Chemical Potential & Number Conservation

particle number often *conserved*
$\rightarrow n = n_{\text{cons}}$ fixed by initial conditions, not T

if particle number conserved, then $\mu \neq 0$ and μ determined by solving $n_{\text{cons}} = n(\mu, T) \rightarrow \mu(n_{\text{cons}}, T)$

so: $\boxed{\mu \neq 0 \ \Leftrightarrow \text{particle number conservation}}$

Chemical Potential and Reactions

reactions change particle numbers among species

in "chemical" equilibrium: forward rate = reverse rate
for example: "two-to-two" reaction $a + b \leftrightarrow A + B$

conservation laws (charge, baryon number, etc.)
force relations between chemical potentials
so in above example: $\mu_a + \mu_b = \mu_A + \mu_B$
sum of chemical potentials "conserved"

in general:

$$\sum_{\text{initial particles}\, i} \mu_i = \sum_{\text{final particles}\, f} \mu_f \tag{9}$$

Equilibrium Thermodynamics

Gas of mass m particles at temp T:
n, ρ, and P in general complicated
because of $E(p) = \sqrt{p^2 + m^2}$
but simplify in ultra-rel and non-rel limits

Non-Relativistic Species

$E(p) \simeq m + p^2/2m$, $T \ll m$
for $\mu \ll T$: Maxwell-Boltzmann, same for Boson, Fermions

for non-relativistic particles = *matter*
energy density, number density vs T?
Q: recall $n(a), \rho(a)$ *and* $T(a)$*?*

Non-Relativistic Species

number density

$$n = \frac{g}{(2\pi\hbar)^3} e^{-(mc^2-\mu)/kT} \int d^3p \; e^{-p^2/2mkT} \tag{10}$$

$$= g e^{-(mc^2-\mu)/kT} \left(\frac{mkT}{2\pi\hbar^2}\right)^{3/2} \tag{11}$$

energy density:

$$\rho c^2 = \langle En \rangle = \varepsilon_{\text{rest mass}} + \varepsilon_{\text{kinetic}} \tag{12}$$

$$= mc^2 \, n + \frac{3}{2} \, kT \, n \tag{13}$$

$$\simeq \varepsilon_{\text{rest mass}} = mc^2 n \tag{14}$$

pressure

$$P = \frac{\langle pvn \rangle}{3} = \frac{\left\langle p^2 n/m \right\rangle}{3} = \frac{2}{3} \varepsilon_{\text{kinetic}} \tag{15}$$

$$= nkT \ll \rho c^2 \tag{16}$$

recover the ideal gas law!

The Ratio of Baryons to Photons

The number of barons per photon
is the "baryon-to-photon ratio" $\boxed{\eta \equiv n_B/n_\gamma}$

photons not conserved in general:
e.g., Brehmsstrahlung $e \to e + \gamma$
so chem pot $\mu_e = \mu_e + \mu_\gamma \to \mu_\gamma = 0$
$\to n_\gamma \sim T^3$: fixed by T alone

baryons conserved:
#baryons = const in comoving vol
$d(n_B a^3) = 0 \to n_B \propto a^{-3}$
$\to$ so $\mu_B(T) \neq 0$ enforces this scaling

Thus we have

$$\eta = \frac{n_{B,0}a^{-3}}{n_{\gamma,0}(T/T_0)^3} = \left(\frac{T_0}{a\,T}\right)^3 \eta_0 \tag{17}$$

baryon number conservation: $n_{\mathrm{B}} \propto a^{-3}$
thermal photons: $n_\gamma \propto T^3$

so as long as $T \sim 1/a$ then
$\eta = const$! *baryon-to-photon ratio conserved!*
thus we expect $\eta_{\mathrm{BBN}} = \eta_{\mathrm{CMB}} = \eta_0$!

numerically (from BBN, CMB anisot):

$$\eta_0 \sim 6 \times 10^{-10} \ll 1 \tag{18}$$

huge number of photons per baryon! never forget!

but $\rho_B/\rho_\gamma \sim m_B n_B / T n_\gamma \sim \eta m_B / T \neq const$

Recombination: Equilibrium Thermodynamics

dominant cosmic plasma components γ, p, e, H (ignore He, Li)
equilibrium: equal forward and reverse rates for

$$p + e \leftrightarrow \mathrm{H} + \gamma$$

and so chem potentials have

$$\mu_p + \mu_e = \mu_{\mathrm{H}} \tag{19}$$

recall: for non-rel species $n = g(mT/2\pi\hbar^2)^{3/2} e^{-(m-\mu)/T}$
thus we have **Saha equation**

$$\frac{n_e n_p}{n_{\mathrm{H}}} = \frac{g_e g_p}{g_{\mathrm{H}}} \left(\frac{m_e m_p}{m_{\mathrm{H}}}\right)^{3/2} \left(\frac{T}{2\pi\hbar^2}\right)^{3/2} e^{-(m_e+m_p-m_{\mathrm{H}})/T} \tag{20}$$

$$\approx \left(\frac{m_e T}{2\pi\hbar^2}\right)^{3/2} e^{-B/T} \tag{21}$$

where $B \equiv m_e + m_p - m_{\mathrm{H}} = 13.6$ eV introduce “free electron fraction” $X_e = n_e/n_B$

use $n_B = \eta n_\gamma \propto \eta T^3$

from Extras last time: $n_\gamma = 2\zeta(3)/\pi^2\, T^3$, with $\zeta(3) = \sum_1^\infty 1/n^3 = 1.20206\ldots$

and note that $n_p = n_e$ *Q: why?*, so

$$\frac{n_e^2}{n_{\mathrm{H}} n_B} = \frac{X_e^2}{1 - X_e} = \frac{\sqrt{\pi}}{4\sqrt{2}\zeta(3)} \frac{1}{\eta} \left(\frac{m_e}{T}\right)^{3/2} e^{-B/T} \tag{22}$$

Q: sanity checks? what sets characteristic T scale?

Q: when is $X_e = 0$ (exactly)?

At last–recombination!

Q: how define physically?

Q: how define operationally, in terms of X_e?

Q: given some $X_{e,\mathrm{rec}}$, how to get z_{rec}?

Director's Cut Extras

Kinetic Theory of Pressure due to Particle Motions

consider cubic box, sidelength L (doesn't really need to be cubic)
contain "gas" of N particles: can be massive or massless
particles collide with walls, bounce back elastically
particles exert force on wall $\leftrightarrow$ wall on particles
this lead to bulk *pressure*

focus on one particle, and its component of motion
in one (arbitrary) axis x: speed v_x, momentum p_x

- *elastic* collision: $p_{x,init} = -p_{x,fin} \rightarrow \delta p_x = 2p_x$
- collision time interval for same wall: $\delta t_x = v_x/2L$
- single-particle *momentum transfer* (force) per wall:
  $F_x = \delta p_x / \delta t_x = p_x v_x / L$
- *single-particle force* per wall area:
  $P = F_x/L^2 = p_x v_x / L^3 = p_x v_x / V$

Q: total pressure?

total pressure is sum over all particles:

$$P = \sum_{\text{particles}} \sum_{\ell=1}^{N} \frac{p_x^{(\ell)} v_x^{(\ell)}}{V} \tag{23}$$

can rewrite in terms of an average momentum flux

$$P = \frac{N}{V} \frac{\sum_{\ell=1}^{N} p_x^{(\ell)} v_x^{(\ell)}}{N} = \langle p_x v_x \rangle \, n \tag{24}$$

where $n = N/V$ is *number* density
$\langle p_x \rangle \, n$ would be average *momentum density* along x
and $\langle p_x v_x \rangle \, n$ is average *momentum flux* along x

if particle gas has isotropic momenta, then

$$\langle p_x v_x \rangle = \langle p_y v_y \rangle = \langle p_z v_x \rangle = \frac{1}{3} \langle \vec{p} \cdot \vec{v} \rangle = \frac{1}{3} \langle pv \rangle \tag{25}$$

so $\boxed{P = \frac{1}{3} \langle pv \rangle \, n}$

Ultra-Relativistic Species

$E(p) \simeq cp \gg mc^2$ (i.e., $kT \gg mc^2$):
Also take $\mu = 0$ ($\mu \ll kT$)

energy density, number density?
Q: recall the answers?

for relativistic bosons

number density

$$
\begin{aligned}
n_{\mathrm{rel,b}} &= \frac{g}{(2\pi\hbar)^3}\int d^3p\ \frac{1}{e^{cp/kT}-1} \\
&= \frac{4\pi g}{(2\pi\hbar)^3}\int dp\ p^2\ \frac{1}{e^{cp/kT}-1} = \frac{g}{2\pi^2}\left(\frac{kT}{\hbar c}\right)^3 \int_0^\infty du\ u^2\ \frac{1}{e^u-1} \\
&= g\frac{\zeta(3)}{\pi^2}\left(\frac{kT}{\hbar c}\right)^3 \propto T^3
\end{aligned}
$$

where

$$
\zeta(3) = \sum_{n=1}^{\infty}\frac{1}{n^3} = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots = 1.20206\ldots \qquad (26)
$$

relativistic fermions:

$$n_{\rm rel,f} = \frac{3}{4} n_{\rm rel,b} \tag{27}$$

so $n \propto T^3$ for both
e.g., CMB today: $n_{\gamma,0} = 411\ \mathrm{cm}^{-3}$

energy density: relativistic bosons

$$\begin{aligned}
\rho_{\rm rel,b} c^2 &= \frac{g}{(2\pi\hbar)^3} \int d^3p \ cp \ \frac{1}{e^{cp/kT} - 1} \\
&= \frac{g}{2\pi^2} \frac{(kT)^4}{(\hbar c)^3} \int_0^\infty du \ u^3 \ \frac{1}{e^u - 1} \\
&= g \frac{\pi^2}{30} \frac{(kT)^4}{(\hbar c)^3}
\end{aligned}$$

and for fermions

$$\rho_{\rm rel,f} = \frac{7}{8} \rho_{\rm rel,b} \tag{28}$$

so $\rho \propto T^4$ for both

pressure

$$P_{\text{rel}} = \left\langle \frac{pv}{3} n \right\rangle = \frac{1}{3} \rho_{\text{rel}} c^2 \tag{29}$$

since $v = c$

$P \propto T^4$

Temperature Evolution

If in therm eq, maintain photon occ. #

$$f(p) = \frac{1}{e^{p/T} - 1} \tag{30}$$

but $cp = h\nu = hc/\lambda \propto 1/a(t)$:
$\Rightarrow p = p_0/a$

w/o interactions, const # γ per mode p
$\Rightarrow f(p) = const$
$\Rightarrow p(t)/T(t) = p_0/T_0$
$\Rightarrow \boxed{T/T_0 = p/p_0 = 1/a = 1 + z}$
e.g., at $z = 3$, CMB $T = 4T_0 \simeq 11$ K
(measured in QSO absorption line system!)

recall: used $w = 1/3$ to show $\rho_\gamma \propto a^{-4}$
but blackbody $\rho_\gamma \propto T^4$
together $T \propto 1/a$ (OK!)