# Astro 507
# Lecture 32
# April 14, 2014

Announcements:

- **Preflight 6 (last one!) due Friday 9am**
- yet another awesome cosmology bigshot talk tomorrow:
  Astronomy Colloquium, 4pm Tuesday April 14
  Nick Gnedin, Fermilab and U. Chicago
  "Simulating Reionization: Yesterday, Today, Tomorrow"

Last time: inflation perturbed

*Q: quantum mechanics of inflaton field $\phi$?*

*Q: "confinement" region for $\phi$?*

*Q: mean value of $\langle \phi \rangle$? $\langle \delta\phi \rangle$?*

*Q: what is fate of fluctuation born at comoving scale $\lambda_{\rm com}$?*

*Q: inflation perturbations vs Hawking radiation?*

# Inflation and Quantum Fluctuations

decompose $\phi(\vec{x},t) = \phi_{\text{cl}}(t) + \delta\phi(\vec{x},t)$
with $\langle\phi\rangle = \phi_{\text{cl}}$ and $\langle\delta\phi\rangle = 0$
but quantun fluctuations have $\left\langle(\delta\phi)^2\right\rangle \neq 0$

causal physics operates on scales inside
the (comoving) Hubble length $d_{H,\text{com}} = 1/aH$

so inflaton field effectively "confined" to $\delta x_{\text{com}} \sim d_{H,\text{com}}$
$\rightarrow$ expect quantum energy fluctuation $\Delta E \sim c\Delta p \sim \hbar/d_H \sim \hbar H$

quantum mechanics generates inflaton perturbations

- in static universe, these average to zero
- but during inflation, $H \approx const$ and $a \approx e^{Ht}$
- when fluctuation of scale $\lambda_{\text{com}} = 1/k_{\text{com}} > d_{H,\text{com}}$
  "leaves horizon" and becomes "frozen in" as real perturbation
- comoving Hubble length $d_{\text{H,com}} \propto e^{-Ht}$ shrinks
  ever smaller scales leave horizon

# Evolution of Quantum Perturbations

Write spatial fluctuations in inflaton field
as sum (integral) of Fourier modes:

$$\delta\phi(t,\vec{x}) = \sum_{\vec{k}} \delta\phi_{\vec{k}}(t) e^{i\vec{k}\cdot\vec{x}_{\rm com}} \qquad (1)$$

where $k = k_{\rm com} = 2\pi/\lambda_{\rm com}$ is comoving wavenumber

classical part of $\delta\phi_{\vec{k}}$ inflated away
but quantum part crucial, persists during inflation

in Director's Cut notes:

• inflaton field begins in vacuum state

• evolves as a *quantum harmonic oscillator*

$\rightarrow$ dominated by *vacuum=ground state*

*Q: wavefunction of ground state harmonic oscillator?*

*Q: probability of finding particle at $x$?*

*Q: implications for inflaton fluctuations?*

# Inflation Spectrum
# Statistical Properties

⋆ Recall: inflaton quantum modes $\leftrightarrow$ harmonic oscillator
dominated by vacuum $\leftrightarrow$ ground state $\|\psi_{\text{sho}}(x)\|^2 \sim e^{-x^2/2\Delta x^2}$
$\phi_k \leftrightarrow x$ fluctuations are statistically Gaussian
i.e., perturbations of all sizes occur, but
probability of finding perturbation of size $\delta(R)$
on scale $R$ is distributed as a Gaussian

⋆ inflaton perturbations $\rightarrow$ reheating
$\rightarrow$ radiation, matter perturbations
same levels in both: "adiabatic"



⋆⋆⋆⋆⋆ All of these are bona fide *pre*dictions of inflation
and are testable! *Q: how?*

# Slow Roll and Scale Dependence

Last time, and in Extras today:
dimensionless fluctuation amplitude (variance)
at comoving wavenumber $k = k_{\rm com}$

$$\Delta^2(k) \sim \left(\frac{\delta\rho}{\rho}\right)_k \sim \left(\frac{H^2}{m_{\rm pl}^2}\right)\left(\frac{H}{\dot{\phi}}\right)^2_{aH=k} \sim \left(\frac{V}{\epsilon m_{\rm pl}^4}\right)_{aH=k} \quad (2)$$

evaluated at "horizon crossing" $aH = k$

*Q: how does $aH$ change during inflation?*
*Q: for slow roll, how does $\Delta^2(k)$ change with scale?*

## Inflation Spectrum
## Slightly Tilted Scale Invariance

recall: perturbation leaving horizon have very similar amplitude during inflation $\rightarrow$ nearly same for all lengthscales $\leftrightarrow k$
perturbation rms amplitude

$$\delta^2_{\rm inf}(k) \propto k^{-6\epsilon+2\eta} \tag{3}$$

- ⋆ successful inflation $\Leftrightarrow$ slow roll $\Leftrightarrow \epsilon, \eta \ll 1$ demands
  **perturbation spectrum nearly independent of scale**
  nearly "self-similar," without characteristic scale
  *"Peebles-Harrison-Zel'dovich"* spectrum

- ⋆ successful inflation must end $\rightarrow \epsilon, \eta \neq 0$
  demands small departures from scale-invariance
  **"tilted spectrum"**

# Gravity Waves: Tensor Perturbations

⋆ so far: only looked at density (scalar) perturbations
but also tensor perturbations → gravity waves!

what's really going on: *cosmic metric* is perturbed
spatial part (in a particular coordinate system = gauge):
• unperturbed = FLRW

$$d\ell^2|_{\mathrm{FLRW}} = a(t)^2\ (dx^2 + dy^2 + dz^2) = a(t)^2\ \delta_{ij}\ dx_i\ dx_j \tag{4}$$

with perturbations

$$d\ell^2|_{\mathrm{pert}} = a(t)^2\ e^{2\zeta}\ \gamma_{ij}\ dx_i\ dx_j \tag{5}$$

with *curvature perturbation* the *scalar* function $\zeta(\vec{x}, t)$
∞ *Q: what it its physical effect?*

perturbed metric

$$d\ell^2|_{\text{pert}} = a(t)^2 \; e^{2\zeta} \; \gamma_{ij} \; dx_i \; dx_j \qquad (6)$$

*curvature perturbation* scalar function $\zeta(\vec{x}, t)$ changes local volume

$\rightarrow$ locally: isotropic stretching

*tensor perturbation* is, to lowest order

$$\gamma_{ij} \approx \begin{pmatrix} 1 + h_+ & h_\times & 0 \\ -h_\times & 1 - h_+ & 0 \\ 0 & 0 & 1 \end{pmatrix} = \delta_{ij} + \begin{pmatrix} h_+ & h_\times & 0 \\ -h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (7)$$

with *two independent modes* of amplitude $h_+, h_\times$

*Q: physical effect of these modes?*

*tensor perturbation* is, to lowest order

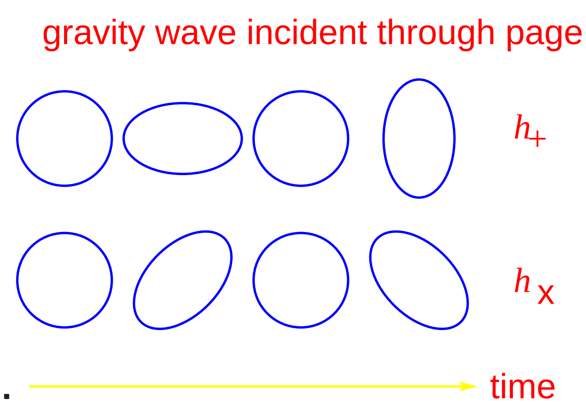
$$\gamma_{ij} \approx \delta_{ij} + \begin{pmatrix} h_{+} & h_{\times} & 0 \\ -h_{\times} & -h_{+} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{8}$$

looks like rotation: roughly speaking preserves volume but changes angles

moreoever: $h$ satisfies massless wave equation!

$h \Leftrightarrow$ **gravitational radiation**

effect on a ring of test particles:



# Metric Flucuations

tensor perturbations directly are metric perturbation
what about the inflaton perturbations?

curvature perturbation in an invariant (coordinate independent):

$$\zeta = \Phi + H\delta t = \Phi + H\frac{\delta\phi}{\dot{\phi}} \tag{9}$$

$\Phi(\vec{x}, t)$ is local *gravitational potential perturbation*

inflation fluctuations $\phi$ also are metric perturbations
but amplitude differs from gravity wave amplitude
by factor $H/\dot{\phi}$
and thus scalar perturbation variance differs by factor

$$r = \frac{\Delta_h^2}{\Delta_\Phi^2} \sim \left(\frac{\dot{\phi}}{H}\right)^2 \sim \epsilon \tag{10}$$

# Inflationary Tensor Perturbations

variance as a function of scale (wavenumber)

$$\Delta_h^2(k) \sim \left(\frac{V}{m_{\rm pl}^4}\right)_{aH=k} \tag{11}$$

- evaluated at "horizon crossing" $aH = k$
- **directly probes inflation potential $V(\phi)$!**
- compare to density ("scalar") perturbations: *tensor-to-scalar ratio*

$$r = \frac{\Delta_h^2}{\Delta_\Phi^2} = 16\epsilon \tag{12}$$

- for $\epsilon \ll 1$, expect $r \ll 1$: scalar dominates

## Testing Inflation: Status to Date

test by measuring density fluctuations
and their statistical properties
on various scales at various epochs

CMB at large angles (large scales, decoupling)
- nearly scale invariant! woo hoo! (COBE 93)
- Gaussian distribution (COBE, WMAP)
  **`www: 3-yr WMAP`** $T$ **`distribution`**
  or nearly so...see Yadav & Wandelt (2007)
- WMAP, Planck: evidence for tilt! favors large scales ("red")!
  Planck (2013): $\alpha = -0.0397 \pm 0.0073$ nonzero at $\sim 5\sigma$!

These did not have to be true!
Not guaranteed to be due to inflation
but very encouraging nonetheless

# Inflation Scorecard

Summary:
Inflation designed to solve horizon, flatness, smoothness
does this, via accelerated expansion driven by inflaton

But unexpected bonus: structure
inflaton field has quantum fluctuations
imprinted before horizon crossing
later return as density fluctuations
$\rightarrow$ inflationary seeds of cosmic structure?!

Thus far: observed cosmic density fields
have spectrum, statistics as predicted by inflation

As of March 17, 2014: gravity wave background too (?!)
probed by CMB polarization!
*all eyes on other polarization experiments!*

# Director's Cut Extras

# Fluctuation Spectrum: In More Detail

Starting point of more rigorous treatment
in equation of motion

$$\ddot{\phi} + 3H\dot{\phi} - \nabla^2\phi + V'(\phi) = 0 \tag{13}$$

write field as sum

$$\phi = \phi_{\text{classical}}(t) + \delta\phi(t, \vec{x}) \tag{14}$$

- **classical amplitude** $\phi_{\text{cl}}(t)$
  spatially homogeneous: *smooth, classical, background* field
  evolves according to classical equation of motion
  $\rightarrow$ this has been our focus thus far; now add
- **quantum fluctuations** $\delta\phi(t, \vec{x})$
  these can vary across space and with time

decompose spatial part of fluctuations into plane waves

$$\delta\phi(t,\vec{x}) = \sum_{\vec{k}} \delta\phi_{\vec{k}}(t) e^{i\vec{k}\cdot\vec{x}_{\rm com}} \tag{15}$$

convenient to label Fourier modes by
*comoving* wavelength $\lambda \equiv \lambda_{\rm com}$, wavenumber $k \equiv k_{\rm com} = 2\pi/\lambda_{\rm com}$
but *physical* wavelength $\lambda_{\rm phys} = a\lambda_{\rm com}$, wavenumber $k_{\rm phys} = k/a$

as long as quantum perturbations $\delta\phi$ small (linear evolution)
each wavelength–i.e., scale–evolves independently
$\rightarrow$ main reason to use Fourier modes

classically $\delta\phi = (\delta\phi)^2 = 0$ by definition!
*Q: what is physical significance of quantum excitations in $\phi$?*

# The Quantum Inflaton Field

quantum mechanically:

- true $\phi$ has fluctuations around background value
- each $\vec{k}$ mode $\leftrightarrow$ independent quantum states (oscillators)
- mode fluctuations *quantized* $\rightarrow$ quanta are inflaton particles analogous to photons as EM quanta
- occupation numbers: $n_{\vec{k}} > 0 \rightarrow$ real particles present
- if $n_{\vec{k}} = 0 \rightarrow \langle \delta\phi \rangle = 0$ no particles (vacuum/ground state) but zero-point fluctuations still present $\left\langle \delta\phi^2 \right\rangle \neq 0$

# Fluctuation Lagrangian

expand each $\vec{k}$ mode around classical value

$$\mathcal{L}_{\vec{k}} = \frac{1}{2}\delta\dot{\phi}_{\vec{k}}^2 - \frac{1}{2}\frac{k^2}{a^2}\delta\phi_{\vec{k}}^2 - \frac{1}{2}V''(\phi_{\rm cl})\delta\phi_{\vec{k}}^2 - V(\phi_{\rm cl}) \quad (16)$$

$$\approx \frac{1}{2}\delta\dot{\phi}_{\vec{k}}^2 - \frac{1}{2}\frac{k^2}{a^2}\delta\phi_{\vec{k}}^2 \quad (17)$$

where slow roll $\to$ potential terms small
$\to$ a **massless simple harmonic oscillator**

$\delta\phi$ vacuum state: zero point fluctuations
formally same a quantum harmonic oscillator!
for *each $k$ mode*, fluctuation *amplitudes random*
but probability distribution is like $n = 0$ oscillator

$$P(\delta\phi_{\vec{k}}) \propto e^{-\delta\phi_{\vec{k}}^2/2\sigma_{\vec{k}}^2} \quad (18)$$

where variance $\sigma_{\vec{k}}^2 = \left\langle \delta\phi_{\vec{k}}^2 \right\rangle$
$\to$ vacuum fluctuation amplitudes have *gaussian* distribution

Total $\phi$ energy density is $\rho_\phi = \rho_{\text{background}} + \rho_{\text{zeropoint}} + \rho_{\text{particles}}$
pre-inflation: could have $\rho_{\text{particles}} \neq 0$
in fact: if thermalized, $\rho_{\text{particles}} \propto T^4$ (radiation)
$\rightarrow$ inflation only begins when $\rho_{\text{background}} \gg \rho_{\text{particles}}$
*Q: what happens to inflatons after inflation begins?*

after inflation begins, universe rapidly expanded, cooled
inflatons diluted away
$\rightarrow$ inflation field driven to **vacuum (ground) state**

Since $\phi$ in quantum vacuum state: fluctuations are zero-point
$\rightarrow$ **gaussian distribution** of amplitudes in each $k$ mode
$\rightarrow$ strong prediction of slow-roll inflation

now want to solve for size of rms $\sigma_k$ at each mode

classically, perturbations have equation of motion

$$\frac{d^2}{dt^2}\delta\phi + 3H\frac{d}{dt}\delta\phi + \frac{k^2}{a^2}\delta\phi + V''\delta\phi \;=\; 0 \qquad (19)$$

$$\frac{d^2}{dt^2}\delta\phi + 3H\frac{d}{dt}\delta\phi + \frac{k^2}{a^2}\delta\phi \approx 0 \qquad (20)$$

(in slow roll: $V''$ term negligible)

# Sketch of Quantum Treatment

Promote $\delta\phi \to$ operator $\widehat{\delta\phi}$
plane wave expansion: $\widehat{\delta\phi} = \sum_{\vec{k}} \widehat{\delta\phi}_{\vec{k}}$
introduce annihilation, creation operators $\hat{a}_{\vec{k}}$, $\hat{a}^{\dagger}_{-\vec{k}}$, then

$$\widehat{\delta\phi}_{\vec{k}} = w_k(t)\,\hat{a}_{\vec{k}} + w_k^*(t)\,\hat{a}^{\dagger}_{-\vec{k}} \tag{21}$$

where $w_k(t)$ is a solution of field equation

$$\ddot{w}_k + 3H\dot{w}_k + \left(\frac{k}{a}\right)^2 w_k = 0 \tag{22}$$

Compare limits:

- $k/a \gg H \to k \gg aH \to \lambda \ll 2\pi d_{H,\mathrm{com}}$
  *Q: physical interpretation of limit?*
  $w_k$ evolves as harmonic oscillator (free massless field)
- $k/a \ll H \to k \ll aH \to \lambda \gg 2\pi d_{H,\mathrm{com}}$
  *Q: physical interpretation of limit?*
  $\dot{w}_k \propto a^{-3} \to 0 \to w_k$ value "frozen"

# Inflation Perturbations: Evolution and Horizons

**sub-horizon scales** $\lambda \ll 2\pi d_{H,\mathrm{com}}$
inflaton fluctuations $\delta\phi$ are causally connected
evolve like harmonic oscillator $\rightarrow$ rms amplitude $\left\langle |w_k|^2 \right\rangle$ *constant*

but cosmic acceleration during inflation $\rightarrow$ $d_{H,\mathrm{com}}$ *shrinks*
since $\dot{d}_{H,\mathrm{com}} = d(aH)^{-1}/dt = d(\dot{a}^{-1})/dt = -\ddot{a}/\dot{a}^2 < 0$ during inf
$d_{H,\mathrm{com}}$ shrinkage: initially sub-horizon scales $\rightarrow$ super-horizon

**super-horizon scales** $\lambda \gg 2\pi d_{H,\mathrm{com}}$
fluctuations out of causal contact
amplitude "frozen in" until post-inflation $d_{H,\mathrm{com}}$ regrows

# Inflation Perturbations: Spectrum of Amplitudes

examine fluctuations from vacuum
$\rightarrow$ find expected amplitudes $w_k$

since fluctuations have *quantum* origin
- cannot predict definite values for mode amplitudes, phases
- but *can* predict statistical properties

for *different* modes $\vec{k}$ and $\vec{k}'$,
*Q: what do we expect?*

for *different* modes $\vec{k}$ and $\vec{k}'$,
expectation is

$$\langle \widehat{\delta\phi}_{\vec{k}} \widehat{\delta\phi}_{\vec{k}'} \rangle = w_k w_{k'} \left\langle \hat{a}_{\vec{k}} \hat{a}^{\dagger}_{\vec{k}'} \right\rangle + \text{c.c.} = 0 \tag{23}$$

because $\left\langle \hat{a}_{\vec{k}} \hat{a}^{\dagger}_{\vec{k}'} \right\rangle = \left\langle \hat{a}_{\vec{k}} \right\rangle \left\langle \hat{a}^{\dagger}_{\vec{k}'} \right\rangle = 0$
$\Rightarrow$ modes are statistically independent
note: true even if $|\vec{k}| = |\vec{k}'| = k$ but $\vec{k} \cdot \vec{k}' = 0$
i.e., different directions $\vec{k} = k\hat{x}, \vec{k}' = k\hat{y}$
$\Rightarrow$ phase $e^{i\vec{k}\cdot\vec{x}}$ is random

for a single mode $k$, vacuum expectation is

$$\begin{aligned} \langle \widehat{\delta\phi}^2_{\vec{k}} \rangle &= |w_k|^2 \left\langle \hat{a}\hat{a}^{\dagger} + \hat{a}^{\dagger}\hat{a} \right\rangle = |w_k|^2 \neq 0 & (24) \\ &= \frac{H^2}{2L^3 k^3} & (25) \end{aligned}$$

where last expression
- from full quantum calculation, in box of size $L$
- to be evaluated at horizon crossing: $k_{\text{phys}} = H \to k = aH$

each in phase space volume

$$d^3xd^3k = \frac{1}{(2\pi L)^3}4\pi k^2 dk = \frac{4\pi k^3}{(2\pi L)^3}\frac{dk}{k} \tag{26}$$

then fluctuation amplitude is

$$P_\phi(k)\frac{dk}{k} \equiv \frac{4\pi k^3}{(2\pi L)^3}|\delta\phi_k|^2\frac{dk}{k} = \left(\frac{H}{2\pi}\right)^2\frac{dk}{k} \tag{27}$$

and so the phase space fluctuation density in $\phi$ is

$$P_\phi(k) = \left(\frac{H}{2\pi}\right)^2_{k=aH} \tag{28}$$

as before, but now

- explicitly seen independence of $k$
- know when to evaluate: at horizon crossing $k = aH$

# Fluctuation Evolution and Spectrum

consider some fixed (comoving) scale $\lambda = 2\pi/k$ key idea: causal physics acts until $\lambda > d_{\mathrm{H,com}}$: "horizon crossing"
$\rightarrow$ quantum fluctuations laid down while inside $d_{\mathrm{H,com}}$
"frozen in" once outside of $d_{\mathrm{H,com}}$

from last time: quantum analysis gives fluctuation variance

$$\left\langle (\delta\phi_k)^2 \right\rangle = \left(\frac{H}{2\pi}\right)^2_{k=aH} \tag{29}$$

to be evaluated at horizon crossing: $k = 1/d_{\mathrm{H,com}} = aH$

# Fluctuation Evolution and Spectrum

consider some fixed (comoving) scale $\lambda = 2\pi/k$ key idea: causal physics acts until $\lambda > d_{\mathrm{H,com}}$: "horizon crossing"
$\rightarrow$ quantum fluctuations laid down while inside $d_{\mathrm{H,com}}$
"frozen in" once outside of $d_{\mathrm{H,com}}$

from last time: quantum analysis gives fluctuation variance

$$\left\langle (\delta\phi_k)^2 \right\rangle = \left(\frac{H}{2\pi}\right)^2_{k=aH} \tag{30}$$

to be evaluated at horizon crossing: $k = 1/d_{\mathrm{H,com}} = aH$

# Inflationary Density Perturbations: Spectrum

Recall: density fluctuations $\rightarrow$ start inflating earlier (later)
$\rightarrow$ more (less) expansion than average
extra scale factor boost $\delta a/a = H\delta t = H\delta\phi/\dot{\phi}$ $\rightarrow$ larger volume
$\rightarrow$ density perturbations have mean square

$$\delta^2_{\mathrm{inf}}(k) \equiv \left(\frac{\delta\rho}{\rho}\right)^2_k \quad (31)$$

$$\sim \left(\frac{\delta a}{a}\right)^2 = \left(\frac{H}{\dot{\phi}}\right)^2 (\delta\phi)^2 = \left(\frac{H}{\dot{\phi}}\right)^2 \left(\frac{H}{2\pi}\right)^2 \quad (32)$$

evaluated at $aH = k$

slow roll: $H$, $\dot{\phi}$ slowly varying
$\rightarrow$ expect fluctuation amplitude $\sim H^4/\dot{\phi}^2 \sim const$
over wide range of $k$

In particular: slow roll $\dot{\phi} = -3V'/H$,
and $H^2 = V/3m_{\rm pl}^2$, which gives

$$\delta_{\rm inf}^2(k) = \frac{1}{12\pi^2 m_{\rm pl}^6}\left(\frac{V^3}{V'^2}\right) = \frac{1}{24\pi^2 m_{\rm pl}^4}\left(\frac{V}{\epsilon}\right) \quad (33)$$

where $\epsilon = m_{\rm pl}(V'/V)^2/2$

anticipating $\sim$ power law behavior,
define $\delta_{\rm inf}^2(k) \sim k^{\alpha(k)}$
then scale dependence is

$$\alpha(k) = \frac{d\ln\delta_{\rm inf}^2(k)}{d\ln k} \quad (34)$$

evaluated when comoving scale $k = aH$ crosses horizon
i.e., this relates $k$ to homogeneous $a$, $\phi$ values

Underlying physical question:
how do cosmic properties–e.g., $H, \rho \approx V$–*change*
while the universe inflates as it slowly rolls?

- if no change $\rightarrow \dot{\phi} = 0 \rightarrow$ same $V, H$ always $\rightarrow \epsilon = 0$
  all scales see same U when leaving horizon $k = aH$
  $\rightarrow$ all scales have same quantum fluctuations
- but *slow* roll $\neq$ *no* roll!
  $\dot{\phi} \neq 0 \rightarrow$ U properties *do* change

recall: $\delta^2_{\rm inf}(k) \propto V/\epsilon$
and as slowly roll $\rightarrow V$ decreasing, $\epsilon$ increasing
and horizon scale $d_{H,\rm com}$ also decreases
*Q: so which scales get larger perturbations? smaller?*

because $V$ decreasing, $\epsilon$ increasing

$\delta^2_{\text{inf}}(k) \propto V/\epsilon$ decreases with time

$\rightarrow$ smaller perturbations later in slow roll

since horizon scale $d_{H,\text{com}}$ decreases

later times $\leftrightarrow$ smaller scales

$\Rightarrow$ slow roll $\rightarrow$ *smaller* perturbations on *smaller* scales

$\Rightarrow$ perturbation spectrum *tilted* to large scales $\rightarrow$ small $k$

in slow roll, $k = aH$ change mostly due to a:

$$d \ln k \approx d \ln a = \frac{da}{a} = H \ dt \tag{35}$$

recast in terms of inflaton potential

$$= \frac{H d\phi}{\dot{\phi}} = -3 \frac{H^2}{V'} d\phi \tag{36}$$

and so

$$\frac{d}{d \ln k} = -m_{\mathrm{pl}}^2 \frac{V'}{V} \frac{d}{d\phi} \tag{37}$$

Using this, can show:

$$\alpha(k) = \frac{d \ln \delta_{\mathrm{inf}}^2(k)}{d \ln k} = -6\epsilon + 2\eta \tag{38}$$

i.e., perturbation spectrum $\delta_{\mathrm{inf}}^2(k) \propto k^{-6\epsilon+2\eta}$

Major result!

*Q: why? what does this mean physically? for cosmology? for inflation?*