Astro 507

Lecture 34

April 18, 2010

Announcements:

- **Problem Set 6 (penultimate PS!) due next Friday**

Last time: opened our eyes to the *inhomogeneous* universe
began structure formation

Q: why is this a fundamentally statistical problem?

Q: what is cosmological "bias"?

Q: what measures of structure would be more fair and balanced?

Q: how to quantify cosmic structure?

Quantifying Density Fluctuations

Given $\rho(t, \vec{x})$, define

mean (average) density $\langle \rho \rangle = \langle \rho(t, \vec{x}) \rangle = \rho_{\mathrm{FRW}}(t)$

(suppress t hereafter)

density fluctuation $\delta\rho(\vec{x}) = \rho(\vec{x}) - \langle \rho \rangle$

density contrast

$$\delta(\vec{x}) = \frac{\delta\rho}{\rho} = \frac{\rho(\vec{x}) - \langle \rho \rangle}{\langle \rho \rangle} \tag{1}$$

where $\delta \neq \delta_{\mathrm{Dirac}}$!

Q: possible range of δ values?

Q: what is $\langle \delta \rangle$?

Q: how does cosmic expansion affect δ?

key measure of cosmic structure: **density contrast**

$$\delta(\vec{x}) = \frac{\delta\rho}{\rho} \equiv \frac{\rho(\vec{x}) - \langle\rho\rangle}{\langle\rho\rangle} \in (-1, \infty) \tag{2}$$

$$\delta_{\vec{k}} = \frac{1}{V}\int \delta(\vec{x})\ e^{i\vec{k}\cdot\vec{x}}\ d^3\vec{x} \tag{3}$$

where average is over large volume V

Q: what is the order-of-magnitude of the density contrast in this room? of the Galactic ISM?

by definition: $\langle\delta\rangle = \frac{1}{V}\int d^3x\,\delta(\vec{x}) = 0$

would like to study structures on different cosmic lengthscales λ

 Q: how to do this using density contrast?

Spectrum of Density Fluctuations

In (large) volume V write $\delta(\vec{x})$ as Fourier series

$$\delta(\vec{x}) = \sum_{\vec{k}} \delta_{\vec{k}}\ e^{-i\vec{k}\cdot\vec{x}} \rightarrow \frac{V}{(2\pi)^3}\int \delta_{\vec{k}}\ e^{-i\vec{k}\cdot\vec{x}}\ d^3\vec{k} \qquad (4)$$

(last expression is continuum limit as $V \to \infty$)
where Fourier coefficients are

$$\delta_{\vec{k}} = \frac{1}{V}\int \delta(\vec{x}) e^{i\vec{k}\cdot\vec{x}}\ d^3\vec{x} \qquad (5)$$

reality: $\delta(\vec{x})^* = \delta(\vec{x}) \rightarrow \delta^*_{\vec{k}} = \delta_{-\vec{k}}$

Beware!
conventions differ on factors of V, sign of exponential
$\rightarrow$ affects dimensions of δ_k

Fourier mode described by amplitude $|\delta_k|$
and **comoving wavenumbers** $k \equiv k_{\text{comov}} = 2\pi/\lambda_{\text{comov}}$
and $\vec{x}$ is comoving as well
physical values are $d\vec{x}_{\text{phys}} = a(t)d\vec{x}$, $\vec{k}_{\text{phys}} = \vec{k}/a(t)$

Q: what is $\delta_{\vec{k}=0}$?
Q: what is connection between $\delta_{\vec{k}}$ and $\delta_{\vec{k}'}$ if $|\vec{k}| = |\vec{k}'| = k$?

Q: how compute a typical value of $\delta\rho/\rho$?
what is it for scale k?

Fun Fourier Facts

$$\delta_{\vec{k}=0} = \int d^3\vec{x}\ \delta(\vec{x}) = \langle\delta\rangle = 0 \tag{6}$$

by definition!

but deeper reason: small $k \leftrightarrow$ large λ

$k \to 0$ is $\lambda \to \infty$ = whole universe

on largest scales, U better be homogeneous!

so: *cosmological principle demands* $\delta_{\text{small}\,k} \to 0$

For $|\vec{k}| = |\vec{k}'| = k$, i.e., same mag, different direction

must find same amplitude fluctuations

...else have a preferred direction*

so cosmo principle $\to$ $\delta_{\vec{k}} = \delta_k$

i.e., wavelength is all that counts k magnitude

* In fact, would $\vec{k}$ anisotropy would manifest not as preferred direction in structure distribution in real space, but rather as preferred *orientation* of structures! (thanks to Z. Lukic for pointing this out)

The Power Spectrum

Want a measure of "typical" fluctuation size"

$\langle \delta\rho/\rho \rangle = \langle \delta \rangle = 0$ by definition, but $\left\langle (\delta\rho/\rho)^2 \right\rangle = \left\langle \delta^2 \right\rangle \neq 0$

$$\left(\frac{\delta\rho}{\rho}\right)^2 = \int d^3\vec{x}\ \delta(\vec{x})^2 \tag{7}$$

$$= \frac{V^2}{(2\pi)^6} \int d^3\vec{x}\ d^3\vec{k}\ d^3\vec{q}\ \ \delta_{\vec{k}}\,\delta_{\vec{q}}\ e^{-i(\vec{k}+\vec{q})\cdot\vec{x}} \tag{8}$$

$$= \frac{V}{(2\pi)^3} \int d^3\vec{k}\ d^3\vec{q}\ \ \delta_{\vec{k}}\,\delta_{\vec{q}}\ \delta_{\text{Dirac}}(\vec{k}+\vec{q}) \tag{9}$$

$$= \frac{V}{(2\pi)^3} \int d^3\vec{k}\ \ \delta_{\vec{k}}\,\delta_{-\vec{k}} \tag{10}$$

$$= \frac{V}{(2\pi)^3} \int d^3\vec{k}\ |\delta_{\vec{k}}|^2 = \frac{V}{(2\pi)^3} \int d^3\vec{k}\ P(k) \tag{11}$$

where $P(k) = |\delta_k|^2$ is the power spectrum

Rewrite in terms of fluctuations per log interval
in wavenumber dk/k:

$$\left(\frac{\delta\rho}{\rho}\right)^2 = \frac{V}{(2\pi)^3}\int d^3\vec{k}\ P(k) = \frac{4\pi V}{(2\pi)^3}\int dk\ k^2\ P(k) \quad (12)$$

$$= \int \frac{4\pi k^3 P(k) V}{(2\pi)^3}\frac{dk}{k} \quad (13)$$

$$\equiv \int \frac{dk}{k}\left(\frac{\delta\rho}{\rho}\right)^2_k \quad (14)$$

where the variance over interval $\delta k/k = d\ln k \sim 1$ is

$$\left(\frac{\delta\rho}{\rho}\right)^2_k \approx \Delta^2(k) \equiv 4\pi\ k^3\ P(k)\ \frac{V}{(2\pi)^3} \quad (15)$$

dimensionless measure of fluctuations on scale k

∞ power spectrum $P(k) \Leftrightarrow \Delta^2(k)$
central object in structure formation
`www: Observed power spectrum` *Q: what stands out?*

Observed Power Spectrum

Gross features of $P(k)$:

★ fairly simple shape: roughly, broken power law
roughly, $P(k) \sim k^1$ at low k,
then steepening negative slope, approaching k^{-3} at large k
we will want to understand why

★ *break at peak:* $k_{\rm peak} \sim 0.02\ h^{-1}{\rm Mpc}^{-1}$
$\rightarrow$ characteristic scale $\lambda_{\rm peak} = 2\pi/k_{\rm peak} \sim 300$ Mpc comoving
we will want to understand what sets this scale!

Features of $\Delta(k) = \sqrt{\Delta^2(k)}$:

★ $\Delta \gtrsim 1$ at $k \gtrsim 0.03$ Mpc $\rightarrow \lambda \lesssim 20$ Mpc
Q: what does this scale tell us?

★ $\Delta \ll 1$ at small k: U $\rightarrow$ homogeneous on large scales
cosmo principle vindicated! Good guess, Al!

dimensionless density variance:
$\delta\rho/\rho \sim \Delta \gtrsim 1$ at $k \gtrsim 0.03$ Mpc $\rightarrow \lambda \lesssim 20$ Mpc

at larger lengthscales (smaller k):
- $\delta\rho/\rho \ll 1$: perturbations small
- expect nonlinearities to be small

at smaller lengthscales (larger k): opposite regime
- $\delta\rho/\rho \gg 1$: perturbations large
- nonlinearities large!

characteristic lengthscale of nonlinearities today

characteristic mass scale:

$$\delta M(\lambda) \sim M_{\rm avg}(\lambda) \sim \frac{4\pi}{3}\lambda^3\rho_0 \sim 3 \times 10^{15}\ M_\odot \tag{16}$$

$\sim$ galaxy cluster masses
$\rightarrow$ *galaxy clusters are largest nonlinear structures today*

Enough already with definitions
and lists of observations!

This is cosmology, not stamp collecting!

Now tell me how to understand it all!

Theory of Cosmological Perturbations

Treat structure formation as initial value problem

- given *initial conditions*: "seeds"
  i.e., adopt spectrum of primordial density perturbations
  prescription for initial $\rho_i(\vec{x})$, $i \in$ baryons, radiation, DM, DE...
  e.g., inflation: scale invariant, gaussian, adiabatic
- follow *time evolution* of $\rho_i(\vec{x})$–i.e., δ_i for each species i
- compare with observed measures of structure

$\star$ agreement (or lack thereof) constrains primordial seeds
e.g., dark matter, inflation, quantum gravity, ...

We want to describe dynamics of cosmic inhomogeneities
Q: which forces relevant? which irrelevant? which scary?

Dynamics Cosmological Perturbations: Overview

Forces/interactions in perturbed, inhomogeneous universe
involve same cosmic particle/field content
as smooth/unperturbed universe
but: can manifest in new/different ways due to spatial variations

Definitely relevant forces on perturbations

- *gravity*: overdensities have extra attraction
  over that of "background" FRW universe
- *pressure*: baryons have thermal pressure $P = nkT$
  photons exert radiation pressure on baryons pre-decoupling
  pressure *gradients* present, unlike in homog. background

Probably irrelevant forces on perturbations (will ignore)

- neutrino interactions with self, other species
- dark matter non-gravity interactions with self, or other species

Scary forces on perturbations (will ignore for now, but worry about)

- if dark energy is a field ϕ, perturbations $\delta\phi$ exert inhomogeneous *negative* pressure why scary? unknown underlying physics
- magnetic fields $\rightarrow$ pressure, MHD forces why scary? unknown initial conditions (primordial B?)

At minimum: we will want to describe baryons & dark matter as inflationary perturbations grow thru radiation, matter eras $\rightarrow$ *gravity* and photon, baryon *pressure* mandatory schematically:

$$\text{acceleration} = -\text{gravity} + \text{pressure} \tag{17}$$

Q: implications for baryons vs dark matter?

For the species and forces we choose to follow:

Q: how can these be described exactly? approximately?

Q: what formalism needed?