Astro 507
Lecture 35
April 21, 2014

Announcements:

- **Problem Set 6 due Wednesday April 30**

Last time: began theory of structure formation
$\rightarrow$ evolution of perturbations to a FLRW cosmology

At minimum: we will want to describe baryons & dark matter
as inflationary perturbations grow thru radiation, matter eras
$\rightarrow$ *gravity* and photon, baryon *pressure* mandatory
schematically:

$$\text{acceleration} = -\text{gravity} + \text{pressure} \tag{1}$$

Q: implications for baryons vs dark matter?

Dynamics of Cosmological Perturbations: Toolbox

need dynamics of inhomogeneous **“fluids”**
in expanding FLRW background

⋆ full treatment: general relativistic perturbation theory
  mandatory for some results *Q: which?*

⋆ good-enough treatment: Newtonian dynamics is FLRW
  as usual, benefits: intuition & simplicity
  costs: limited range of validity

Newtonian Fluid Dynamics & Self-Gravity

Each cosmic species is "fluid" described by fields

- mass density $\rho(\vec{x}, t)$
- velocity $\vec{v}(\vec{x}, t)$
- pressure $P(\vec{x}, t)$: from equation of state $P = P(\rho, T)$

In Newtonian limit: dynamics governed by

mass conservation (continuity) $\partial_t \rho + \nabla \cdot (\rho \vec{v}) = 0$

Euler: "$F = ma$" $\rho d\vec{v}/dt = \rho \partial_t \vec{v} + \rho \vec{v} \cdot \nabla \vec{v} = -\nabla P - \rho \nabla \Phi$

Note: fixed/non-comoving coords need "*convective derivative*"

$d\vec{v}(\vec{x}, t)/dt = (\partial_t + \dot{x}_i \partial_i)\vec{v} = \partial_t \vec{v} + \vec{v} \cdot \nabla \vec{v}$

Newtonian gravity: inverse square $\rightarrow$ Poisson $\nabla^2 \Phi = 4\pi G \rho$

These are general (albeit Newtonian only)
$\rightarrow$ now apply to the Universe

Linear Theory 0: Newtonian, Non-expanding

consider *static*, uniform (infinite) distribution of matter
and introduce small perturbations

$$\rho(\vec{x}) = \rho_0 \, [1 + \delta(\vec{x})] \quad (2)$$
$$v(\vec{x}) = \vec{u}(\vec{x}) \quad (3)$$
$$\Phi_{\rm grav}(\vec{x}) = \Phi_0 + \Phi_1(\vec{x}) \quad (4)$$

where $\delta \ll 1$, and $\Phi_1, \vec{u}$ "small"

we want: time development of (initially) small perturbations
following Sir James Jeans
many key ideas of full expanding-Universe GR result
already appear here!

Newtonian fluid equations: continuity (mass conservation)

$$\partial_t \rho + \nabla \cdot (\rho \vec{v}) = 0 \quad (5)$$
$$\rho_0 \dot{\delta} + \rho_0 \nabla \cdot \vec{u} \approx 0 \quad (6)$$

Euler ("$F = ma$");

$$\rho d\vec{v}/dt = \rho\partial_t\vec{v} + \rho\vec{v}\cdot\nabla\vec{v} \;=\; -\nabla p - \rho\nabla\Phi \tag{7}$$

$$\rho_0\dot{\vec{u}} \;\approx\; -\rho_0 c_s^2\nabla\delta - \rho_0\nabla\Phi_1 \tag{8}$$

where **adiabatic sound speed** $c_s^2 = \partial p/\partial\rho$

Gravity: Poisson (Gauss' law = inverse square force)

$$\nabla^2\Phi \;=\; 4\pi G\rho \tag{9}$$

$$\nabla^2\Phi_1 \;\approx\; 4\pi G\rho_0\delta \tag{10}$$

note inconsistency=cheat! $\nabla^2\Phi_0 \neq 4\pi G\rho_0$: "*Jeans swindle*"

can combine to single eq for linearized density contrast:

$$\partial_t^2\delta - c_s^2\nabla^2\delta = 4\pi G\rho_0\delta \tag{11}$$

Q: behavior for pressureless fluid? "switched-off" gravity? physical significance? important scales?

Density contrast evolves as

$$\partial_t^2 \delta - c_s^2 \nabla^2 \delta = 4\pi G \rho_0 \delta \tag{12}$$

solutions are of the form

$$\delta(t, \vec{x}) = A e^{i(\omega t - \vec{k}\cdot\vec{x})} \equiv D(t)\ \delta_0(\vec{x}) \tag{13}$$

where $\delta_0(\vec{x}) = e^{-i\vec{k}\cdot\vec{x}}$ is init Fourier amp
and time evolution is set by exponent $\omega(k)$:

$$\omega^2 = c_s^2 k^2 - 4\pi G \rho_0 \equiv c_s^2 (k^2 - k_J^2) = \left(\frac{c_s}{k_J}\right)^2 \left[\left(\frac{\lambda_J}{\lambda}\right)^2 - 1\right] \tag{14}$$

key scale: **Jeans length**

$$k_J = \frac{\sqrt{4\pi G \rho_0}}{c_s} \qquad \lambda_J = \frac{c_s}{\sqrt{G\rho_0/\pi}} \sim c_s \tau_{\rm freefall} \tag{15}$$

associate Jeans mass: $M(\lambda_J/2) = 4\pi/3\ \rho_0 (\pi/k_J)^3 \sim c_s^3/G^{3/2}\rho_0^{1/2}$
Q: physically, what expect for $\lambda < \lambda_J$? $\lambda > \lambda_J$?

perturbation growth $\delta_k(t) = \delta_k(t_0) e^{i\omega t}$, with

$$\omega^2 = c_s^2 k^2 - 4\pi G \rho_0 \equiv c_s^2 (k^2 - k_J^2) \qquad (16)$$

Jeans length $\sim c_s \tau_{\text{freefall}}$: sound travel distance in freefall time
$\rightarrow \lambda/\lambda_J \sim$ number of pressure wave crossings during freefall

if $k > k_J$ so $\lambda < \lambda_J$, small scales: pressure can repel contraction
ω real: oscillations about hydrostatic equlilb

if $k < k_J$ so $\lambda > \lambda_J$, large scales: pressure ineffective
ω imaginary, exponential collapse
runaway perturbation growth $D(t) = e^{\omega t} \sim e^{+t/t_{\text{freefall}}}$
(also an uninteresting decaying mode $e^{-\omega t}$)

Q: but what about expanding Universe?
should grav instability be stronger or weaker?

Linear Theory I: Newtonian Analysis in Expanding U.

Recall: Newtonian analysis legal for small scales, weak gravity
$\rightarrow$ okay for linear analysis inside Hubble length
apply to **matter-dominated U.**

Coordinate choices

Eulerian time-indep grid $\vec{x}$ fixed in physical space
expansion moves unperturbed fluid elts past as $\vec{x}(t) = a(t)\vec{r}$

Lagrangian coords $\vec{r}$ time-indep but expand in physical space
following fluid element: *locally* comoving

$\Rightarrow$ spatial gradients: $\nabla_{\vec{x}} = (1/a)\nabla_{\vec{r}}$

Unperturbed (zeroth order) eqs,
using $\rho_0 = \rho_0(t)$, $\vec{v}_0 = \frac{\dot{a}}{a}\vec{x} = \dot{a}\vec{r}$

$$\partial_t \rho_0 + \nabla \cdot (\rho_0 \vec{v}) = \dot{\rho_0} + \rho_0 \frac{\dot{a}}{a} \nabla_{\vec{x}} \cdot \vec{x} = 0 \quad (17)$$

$$\dot{\rho}_0 + 3\frac{\dot{a}}{a}\rho_0 = 0 \qquad \Rightarrow \rho_0 \propto a^{-3} \quad (18)$$

Poisson:

$$\nabla^2\Phi_0 = \frac{1}{x^2}\partial_x(x\partial_x\Phi_0) = 4\pi G\rho_0 \;\;\Rightarrow\;\; \Phi_0 = \frac{2\pi G\rho_0}{3}x^2 = \frac{2\pi G\rho_0}{3}a^2r^2$$

$$\nabla_{\vec{x}}\Phi_0 = \frac{4\pi G\rho_0}{3}\vec{x} \qquad \nabla_{\vec{r}}\Phi_0 = \frac{4\pi G\rho_0}{3}a\vec{r}$$

Euler

$$d(\dot{a}\vec{r})/dt = \ddot{a}\vec{r} = \frac{\ddot{a}}{a}\vec{x} \;=\; -\frac{4\pi G\rho_0}{3}\vec{x} \tag{19}$$

$$\frac{\ddot{a}}{a} \;=\; -\frac{4\pi G\rho_0}{3} \tag{20}$$

Fried accel; with continuity $\rightarrow$ Friedmann

Zeroth order fluid equations $\rightarrow$ usual expanding U
in non-rel approximation

Now add perturbations $\rho_1 = \rho_0 \delta$, $\vec{v}_1$, Φ_1

simplest to use comoving (Lagrangian) coords
follow observers in unperturbed Hubble flow
perturbation fluid elements $\vec{x}(t) = a(t)\vec{r}(t)$
peculiar fluid velocity $\vec{v}_1(t) = a(t)\vec{u}(t)$

plug in, keep only terms linear in perturbations $(\nabla = \nabla_{\vec{r}})$
$\rightarrow$ *perturbation evolution to first (leading, linear) order*

$$\dot{\vec{u}} + 2\frac{\dot{a}}{a}\vec{u} = -\frac{1}{a^2}\nabla\Phi_1 - \frac{1}{a}\frac{\nabla \delta p}{\rho_0} \quad (21)$$

$$\dot{\delta} = -\nabla \cdot \vec{u} \quad (22)$$

consider the case of $\Phi_1 = 0$ and $\delta p = 0$, but initial $\vec{u} \neq 0$
Q: what does this represent physically? what happens? why?
Q: implications for the situation when $\Phi_1 \neq 0$ and $\delta\rho \neq 0$?

Velocity Perturbation Evolution

peculiar velocity $\vec{v_1} = a(t)\ \vec{u}$ evolves as

$$\dot{\vec{u}} + 2\frac{\dot{a}}{a}\vec{u} = -\frac{1}{a^2}\nabla\Phi_1 - \frac{1}{a}\frac{\nabla\delta p}{\rho_0} \tag{23}$$

if no pressure nor density perturbations
then $\dot{u} = -2Hu$, and so $u \propto 1/a^2$
and physical speed evolves as $v_1 \propto 1/a$

but recall: long ago derived FLRW test particle speed
evolves as $\vec{v}(t) = \vec{v}_0/a(t)$
$\rightarrow$ pressureless fluid's peculiar vel redshifts same as free particle
$\rightarrow$ expansion acts as "drag" on particles

if $\Phi_1, \delta p \neq 0$: Hubble "drag" still present
removes kinetic energy from collasping objects
allows total energy to change (decrease) with time
$\rightarrow$ *binding increases!*

Linearized Density Evolution

now look for plane-wave solutions $\leftrightarrow$ write as Fourier modes
e.g., $\delta(\vec{r}) \sim e^{-i\vec{k}\cdot\vec{r}}$, with $\vec{k}$ comoving wavenumber

$$\ddot{\delta}_k + 2\frac{\dot{a}}{a}\dot{\delta}_k = \left(4\pi G\rho_0 - \frac{c_s^2 k^2}{a^2}\right)\delta_k \tag{24}$$

if no expansion ($a = 1, \dot{a} = 0$), recover Jeans solution

with expansion:

- Hubble "friction" or "drag" $-2H\dot{\delta}$ opposes density growth
- still critical Jeans scale: $k_J = \sqrt{4\pi G\rho_0 a^2/c_s^2}$
  expect oscillations on small scales, collapse on larger

Director's Cut Extras

Correlation Function

Taking $\left\langle \delta(\vec{x})^2 \right\rangle$ gives $(\delta\rho/\rho)^2_{\text{rms}}$

$\rightarrow$ overlap of density contrast with itself

(at same point in space)

What about $\xi(\vec{r}) = \langle \delta(\vec{x})\delta(\vec{x}+\vec{r})\rangle$ (fixed $\vec{r}$, avg over $\vec{x}$)

(two-point or auto-) correlation function

- physical significance?

  what if ρ at each space point independent of all other points?

  opposite case: what if strictly periodic (lattice)?
- sign(s)? meaning of sign(s)?
- behavior at large, small $|\vec{r}|$?
- significance of r at which $\xi(r) = 0$?
- dependence on $\hat{r} = \vec{r}/|\vec{r}|$?

Correlation function: avg of density contrast overlap with itself, "lagged" by spacing $\vec{r}$:

$$\xi(\vec{r}) = \langle \delta(\vec{x})(\vec{x}+\vec{r}) \rangle = \frac{1}{V} \int \delta(\vec{x}) \; \delta(\vec{x}+\vec{r}) \; d^3\vec{x} \tag{25}$$

- physically: given δ somewhere, measures typical δ separated by $\vec{r}$
- if each space point independent of all others, no matter how close, then: $\xi(\vec{r}) = 0$ for $\vec{r} \neq 0$
- but even if this were ever true, local physics *must* remove independence
- since $\delta \in (-1, \infty)$, ξ can be negative (must be for some r!)

Demo–toy model transparencies

Q: if structure in a lattice, what does ξ measure?

Q: what is significance of first zero of ξ?

Correlation function in an idealized "Lattice Universe"

- if lattice of galaxy clusters, ξ oscillates with lattice periodicity
  $\rightarrow$ gives typical cluster size, and typical cluster separation
  true even if not lattice

Correlation function generally:

- first $\xi(\vec{r}) = 0$ gives typical cluster size
- small $\vec{r}$: must have $\xi \rightarrow (\delta\rho/\rho)^2 > 0$
  large $\vec{r}$: correlations must vanish $\xi \rightarrow 0$
  (cosmo principle/horizons)
- isotropy: $\xi(\vec{r}) = \xi(r)$ independent of direction

In Fourier space:

$$\xi(\vec{r}) = \frac{1}{V}\int \delta(\vec{x})\ \delta(\vec{x}+\vec{r})\ d^3\vec{x} \tag{26}$$

$$= \frac{V}{(2\pi)^6}\int \delta_{\vec{k}}\,\delta_{\vec{q}}\,e^{-i(\vec{k}+\vec{q})\cdot\vec{r}}e^{-i\vec{q}\cdot\vec{r}}\quad d^3\vec{k}\ d^3\vec{k}\quad d^3\vec{x} \tag{27}$$

$$= \frac{V}{(2\pi)^3}\int \delta_{\vec{k}}\,P(k)\ e^{-i\vec{k}\cdot\vec{r}}\,d^3\vec{k} \tag{28}$$

$$= \int \Delta^2(k)\ e^{-i\vec{k}\cdot\vec{r}}\ \frac{dk}{k} \tag{29}$$

correlation function is Fourier transf of power spectrum $P(k)$

Q: why observationally useful?

example of general case: $P(k)$ "all you need know"

about density field for Gaussian fluctuations...

Power-Law Spectra

Consider a power-law power spectrum $P(k) \sim k^n$

- useful approximation over large k ranges
- inflation predicts initial conditions of this form
- recall $\Delta^2(k) \sim k^3 P(k) \sim k^{n+3}$
  homogeneity $\rightarrow n > -3$
  also must be cutoff at large k *Q: physical meaning?*

Note: this is only a first approximation
But we will see that the *true* power spectrum
is *not* a power law

- theory predicts deviations ("baryon acoustic oscillations")
- observations have begun to detect these

Rough meaning of n:
for a lengthscale $x \sim \lambda \sim 1/k$,
imagine “filtering” or “smoothing’ density field over this scale
i.e., replace true density at each point with
density averaged over radius x

then for each lengthscale x
corresponding mean mass scale is $M \sim \rho_0 x^3 \sim x^3$
then $(\delta_{\rm rms})^2 \sim \int_0^{1/x} \Delta(k)\, dk/k \sim M^{-(n+3)/3}$
and so root-mean-square mass fluctuation is

$$\delta_{\rm rms} \sim M^{-(n+3)/6} \tag{30}$$

recall: for large k, $P(k) \sim k \to n = 1$
$\to \delta_{\rm rms} \sim M^{-2/3}$ drops for large masses:
approach homogeneity as $M \to \infty$

Correlation Function

if $P(k) \sim k^n$, then ξ also a power law:
$\xi(r) \sim r^{-(n+3)}$; for galaxies

$$\xi_{\rm gal}(r) \simeq \left(\frac{r}{5\ h^{-1}\ \rm Mpc}\right)^{-1.8} \tag{31}$$

where correlation length $r_{\rm corr} = 5\ h^{-1}$ Mpc
sets scale where ξ starts to become small
$\rightarrow$ typical structure size

note SDSS galaxy-galaxy ξ index gives $n \sim -1.2$
consistent with SDSS galaxy-galaxy $P(k)$ measurements
on the same scales (check!)

Filtered Density

Conceptually useful, and observationally practical to imagine "filtering" the density field $\rho(\vec{x})$ over some lengthscale R, mass scale

$$M(R) = \rho_0 V(R) = 1.16 \times 10^{12} h^{-1} \left(\frac{R}{1\, h^{-1}\,\mathrm{Mpc}} \right)^3 M_\odot \qquad (32)$$

$\rightarrow$ gives "smoothed" field at this scale

To implement mathematically, introduce window function weights the neighboring points; simplest is "top hat"

$$W(r;R) = \begin{cases} 1 & r \le R \\ 0 & r > R \end{cases} \qquad (33)$$

using this, we have a "filtered variance"

$$\sigma^2(R) = \int d^3\vec{x}\ \delta(\vec{x})^2\ W(|\vec{x}|;R) \qquad (34)$$

$$= \frac{V}{(2\pi)^3} \int d^3\vec{k} P(k)\ W_k \simeq \Delta^2(k \sim 1/R) \qquad (35)$$

Scale of Nonlinearities Now

Key scale R: where $\sigma^2(R) = 1 \rightarrow$ linear/nonlinear boundary
empirically: near $R \sim 10$ Mpc
i.e., $M \sim 10^{15} M_\odot \rightarrow$ rich clusters!
$\rightarrow$ scale just becoming nonlinear today

key parameter set by convention: σ_8 a.k.a. "sigma-8"

$$\sigma_8^2 \equiv \sigma^2(8\ h^{-1}\,\mathrm{Mpc}) \simeq 0.8 \tag{36}$$

Gaussian Perturbations

So far: compared sizes of perturbations across different scales k
$\rightarrow$ via shape of $P(k) = |\delta_k|^2$

but can also ask: at one fixed scale k
what range of amplitudes δ_k appear?
i.e., sample Fourier amplitude δ_k over
different volumes $V \gg k^{-3}$
each a "realization" of true underlying cosmic sample
$\rightarrow$ what distribution results?

if Fourier mode amplitudes independent
and arise from causally disconnected regions
then central limit theorem ("law of averages")
$\rightarrow$ δ_k Gaussian distributed
$\rightarrow$ this is also prediction from inflation

i.e., for density field smoothed over size R
probability of finding fluctuation amplitude δ is

$$P(\delta) = \frac{1}{\sqrt{2\pi\sigma(R)}} e^{-\delta^2/2\sigma^2(R)} \tag{37}$$

implicitly require $|\delta| \ll 1$ *Q: why*

Observationally: holds as far as we can tell