Astro 507

Lecture 35

April 21, 2014

Announcements:

- **Problem Set 6 due next Wednesday April 30**
- Physics Colloquium today:
  "*Majorana Fermions and Neutrino Mass*"
  Prof. Liang Yang, UIUC Physics
- **ICES** available online – please do it!

Last time: Newtonian perturbation theory

Q: results in non-expanding universe? characteristic speed? timescale? lengthscale?

With cosmic expansion:

Q: what's "peculiar" about a velocity? evolution? implications?

Q: density perturbations: what changes?

Linearized Density Evolution

now look for plane-wave solutions $\leftrightarrow$ write as Fourier modes
e.g., $\delta(\vec{r}) \sim e^{-i\vec{k}\cdot\vec{r}}$, with $\vec{k}$ comoving wavenumber

$$\boxed{\ddot{\delta}_k + 2\frac{\dot{a}}{a}\dot{\delta}_k = \left(4\pi G\rho_0 - \frac{c_s^2 k^2}{a^2}\right)\delta_k} \tag{1}$$

if no expansion ($a = 1, \dot{a} = 0$), recover Jeans solution

with expansion:

- Hubble "friction" or "drag" $-2H\dot{\delta}$ opposes density growth
- still critical Jeans scale: $k_J = \sqrt{4\pi G\rho_0 a^2/c_s^2}$
  expect oscillations on small scales, collapse on larger

Unstable Modes: Matter-Dominated U

Consider **large scales** $\lambda \gg \lambda_J$

$$\ddot{\delta}_k + 2\frac{\dot{a}}{a}\dot{\delta}_k \approx 4\pi G\rho_0\delta_k \tag{2}$$

in *Matter-dominated U:* $8\pi G\rho/3 = H^2 = (2/3t)^{-2} = 4/9t^2$, so

$$\ddot{\delta}_k + \frac{4}{3t}\dot{\delta}_k - \frac{2}{3t^2}\delta_k = 0 \tag{3}$$

Q: how many independent solutions? how to solve?

Matter-dominated U, large scales:

$$\ddot{\delta}_k + \frac{4}{3t}\dot{\delta}_k - \frac{2}{3t^2}\delta_k = 0 \tag{4}$$

eq homogeneous in $t \to$ try *power law solution*

trial $\delta \sim t^s$ works if

$$s(s-1) + 4s/3 - 2/3 = 0 \tag{5}$$

solutions $s = 2/3, -1$:
growing and decaying modes

$$\delta_+(t) = \delta_+(t_i)\left(\frac{t}{t_i}\right)^{2/3} \quad ; \quad \delta_-(t) = \delta_-(t_i)\left(\frac{t}{t_i}\right)^{-1} \tag{6}$$

- growing mode dominates
- Hubble friction: exponential collapse softened to power law

★ Note: solutions indep of k *Q: why a big deal?*

Linear Growth Factor

each unstable Fourier mode grows with time as

$$\delta_k(t) \propto D(t) \sim t^{2/3} \sim a \sim \eta_{\rm conform}^2 \quad (7)$$

growth independent of wavenumber k

- in k-space, all unstable modes grow by same factor and transform to real space, find
- on large scales (but still subhorizon)

$$\delta(t, \vec{x}_{\rm large}) \simeq D(t)\delta(t_i, \vec{x}_{\rm large}) \quad (8)$$

$\Rightarrow$ entire density contrast pattern grows with same amplification:

$\Rightarrow$ **linear grow factor** $D(t)$ applies to whole field

Q: implications for power spectrum? fluctuations at scale R?

on large scales (but still subhorizon)

$$\delta_k(t) = D(t)\delta_k(t_i) \tag{9}$$

thus, on subhorizon scales power spectrum evolves as

$$P(k,t) = |\delta_k(t)^2| = D(t)^2 P_i(k) \sim a^2(t)\ P_i(k) \quad \propto \quad a^2 \tag{10}$$

RMS fluctuation at scale R:

- roughly $(\delta\rho/\rho)_k^2 \sim \Delta^2(k) \sim k^3\ P(k)$
- so scale R, corresponding to $k \sim 1/R$ has RMS fluctuation $\sigma(R) \sim \Delta(k = 1/R)$:

$$\sigma(R,t) \sim \Delta(k = 1/R) \sim a(t)\ \sigma_i(R) \sim \frac{\sigma_i(R)}{1+z} \tag{11}$$

 RMS fluctuations grow as $\sigma \propto a$

Applications to CMB: Naïve Inferences

before decoupling: pressure dominated by photons
$\rightarrow$ expect oscillations – and see them!
after decoupling: growing mode

CMB anisotropies are a snapshot
of perturbations at last scattering
can quantify level: $(\delta T/T)_{\mathrm{ls}} \sim 10^{-5}$ at $z_{\mathrm{ls}} \sim 1100$

But matter has $\rho \propto a^{-3} \propto T^3$, so $\delta\rho/\rho = 3\delta T/T$
$\rightarrow$ $\delta_{\mathrm{obs}}(z = 1100) \sim 3 \times 10^{-5}$ at last scattering
So today, expect fluctuations of size

$$\delta_0 = \frac{D_0}{D_{\mathrm{ls}}}\delta_{\mathrm{ls}} = \frac{a_0}{a_{\mathrm{ls}}}\delta_{\mathrm{ls}} = (1 + z_{\mathrm{ls}})\delta_{\mathrm{ls}} \sim 0.05 \ll 1 \tag{12}$$

Should still be very small–no nonlinear structures, such as us!
Q: obviously wrong–egregiously naïve! What's the flaw?
What's the fix?

Cosmic Diversity: Evolution of Multiple Components

Thus far: *implicitly assumed a baryons-only universe*: not ours!

Cosmic "fluid" contains many different species
with different densities, interactions
baryons, photons, neutrinos, dark matter, dark energy

Each component i has its own equations of motion, e.g.:

$$\ddot{\delta}_i + 2H\dot{\delta}_i = -\frac{c_{s,i}^2 k^2}{a^2}\delta_i + 4\pi G\rho_0 \sum_j \Omega_j \delta_j \tag{13}$$

species interact via pressure, gravity: evolution eqs *coupled*
▷ gravity from dominant Ω drives the other components
 ▷ each species' (pressure) response depends on
microphysics of its interactions, encoded in sound speed $c_{s,i}$

Matter Instability in the Radiation Era

(dark) matter perturbation δ_m during radiation domination

- pick subhorizon scale: growth possible
- focus on $k < k_J$: Jeans unstable (can ignore pressure) and high-k modes just oscillate anyway
- treat radiation perturbations as *smooth*: $\delta_{\rm rad} \approx 0$ $P_r = \rho_r/3$: huge, fast $c_s \sim c$ any perturbations will be oscillatory anyway
- dark matter: weak interactions $\rightarrow$ pressureless $\rightarrow c_s = 0$!

Evolution simple – to rough approximation, for these k:

$$\ddot{\delta}_m + 2\frac{\dot{a}}{a}\dot{\delta}_m \overset{\rm rad-dom}{=} \ddot{\delta}_m + \frac{1}{t}\dot{\delta}_m \approx 0 \tag{14}$$

Simple solutions: growing mode plus decaying mode

$$\delta_m(t) = D(t)\delta_m(t_i) = \left(D_1 \log t + \frac{D_2}{t}\right)\delta_m(t_i) \tag{15}$$

Q: implications? what about baryons?

Found $D(t) \sim D_1 \log t$: "growing" mode hardly grows!

$\star$ dark matter perturbations *frozen* during rad dom
dark matter growth quenched by
→ non-growth of radiation perturbations
→ extra expansion due to radiation

$\star$ *dark matter perturbation growth stalled*
until end of radiation era: **matter-radiation equality**
i.e., $\rho_{\text{matter}} = \rho_{\text{radiation}}$ when $z_{\text{eq}} \sim 3 \times 10^4$
Q: is before or after BBN? recomb?
⇒ this marks onset of structure formation

Q: how does this update our naive CMB calculation?
Hint: then, correct reasoning for $\delta = \delta_b$ only

baryons tightly coupled to photons till recombination
$\rightarrow$ so dark matter perturbations begin growth earlier

And so: DM has grown more! update earlier estimate and focus on dark matter

$$\delta_{m,0} = \frac{D_{\mathrm{ls}}}{D_{\mathrm{eq}}}\delta_{b,0} \sim \frac{1+z_{\mathrm{eq}}}{1+z_{\mathrm{ls}}}\delta_b \sim 30 \times 0.05 \sim 1 \tag{16}$$

DM can grow to nonlinearity today!

⋆ existence of collapsed cosmic structures
requires collisionless dark matter!
⋆ independent argument for large amounts of
weakly interacting matter throughout universe!

CMB Anisotropies

CMB Anisotropies

Between matter-radiation equality and recombination:

- dark matter perturbations grow
  form deepening potential wells
- baryons, electrons tightly coupled to photons (plasma)
  undergo oscillations: *gravity vs pressure = acoustic*

Q: what is the largest scale which can oscillate?
Q: for each mode k, what sets oscillation frequency?
Q: at fixed t, which scales have oscillated the most? the least?
Q: how is this written on the CMB?

Director's Cut Extras

Non-relativistic Cosmic Kinematics

gas particles have random thermal speeds, momenta
how are these affected by cosmic expansion?

Classical picture:
consider non-rel free* particle moving w.r.t. comoving frame
$\vec{\ell}_{\rm com}(t) \neq const$, and so $\vec{\ell}_{\rm phys} = a(t)\ell_{\rm com}(t)$:

$$\begin{array}{rcccc} \vec{v} = d\vec{\ell}_{\rm phys}/dt & = & \dot{a}(t)\ell_{\rm com}(t) & + & a(t)\dot{\ell}_{\rm com}(t) \\ & = & H\vec{\ell}_{\rm phys} & + & \vec{v}_{\rm pec} \\ & = & \text{Hubble flow} & + & \text{peculiar velocity} \end{array}$$

Note that peculiar velocity v is always w.r.t. the comoving frame–i.e., the particle speed compared to that of a stationary fundamental observer *at the same point*

*i.e., except for gravitation

consider a comoving observer at the origin, $\vec{x} = 0$
in time δt, a particle moves w.r.t. comov frame
physical dist $\delta\vec{x}_{\rm phys} = \vec{v}_{\rm pec}\delta t$

but due to Hubble flow, a comoving (fundamental) observer at $\delta\vec{x}_{\rm phys}$ is moving away from the origin at speed $\vec{v}_{\rm com} = H\delta\vec{x}_{\rm phys}$

thus the new speed of the particle relative to its new comoving neighbor is given by the relative velocity
$\vec{v}'_{\rm pec} = \vec{v}_{\rm pec} - \vec{v}_{\rm com}$
(where we used the non-rel velocity addition law)
and so the peculiar velocity *changes* by

$$\delta\vec{v}_{\rm pec} = -H\delta\vec{x}_{\rm phys} = -\frac{\dot{a}}{a}\vec{v}_{\rm pec}\delta t = -\frac{\delta a}{a}\vec{v}_{\rm pec} \tag{17}$$

Q: physical implications?

$\delta v_{\text{pec}}/v_{\text{pec}} = -\delta a/a \Rightarrow$ physical peculiar velocity $v_{\text{pec}} \propto 1/a$:

- $mv_{\text{non-rel}} = p_{\text{non-rel}} = p_0/a$
- comoving peculiar velocity $d\ell_{\text{com}}/dt \propto 1/a^2$
  slowdown w.r.t. comoving frame: velocity "decays"
  not a "cosmic drag" but rather kinematic effect
  due to struggle to overtake receding of cosmic milestones

Quantum picture:

recall for photons, $p_{\text{rel}} = h/\lambda \sim 1/a$ (de Broglie)
but de Broglie holds for matter too: $p_{\text{non-rel}} = h/\lambda_{deB} \sim 1/a$
$\Rightarrow$ again, $p_{\text{non-rel}} = p_0/a$

true in general, now apply to thermal gas non-relativistic gas: Maxwell-Boltzmann

$$n = \frac{g}{(2\pi\hbar)^3} e^{-(mc^2-\mu)/kT} a^{-3} \int d^3p_0 \; e^{-p_0^2/2mka^2T}$$

if occupation number constant (particle conservation)
need $a^2 T(a) = T_0 = const$ and thus $T_{\text{non-rel}} \propto 1/a^2$:

$$T_{\text{non-rel,decoupled}} = \left(\frac{a_{\text{dec}}}{a}\right)^2 T_{\text{decoupling}} = \left(\frac{1+z}{1+z_{\text{dec}}}\right)^2 T_{\text{decoupling}}$$

evaluate for $z_{\text{dec}} = z_{\text{ri}}$: estimate

$$T_{\text{gas,today}} \sim \frac{T_{\gamma,0}}{1+z_{\text{dec,gas}}} \sim 6 \times 10^{-3}\ \text{K} \tag{18}$$

Q: do the experiment...?

Q: what went wrong?

Inhomogeneities: The Spice of Life

So far: we have assumed perfect homogeneity!
If universe strictly homogeneous
indeed would cool to $T_{\rm gas} \ll T_0$

But happily, U. definitely inhomogeneous on small scales!
gravity amplifies density contrast *Q: why?*
"the rich get richer, the poor get poorer"

this allows for motion, condensation of matter
halo formation, mergers, shocks, star formation, quasars, ...
these overdense structures release energy
lead to diversity of cosmic matter and radiation today!

But how did we get the inhomogeneities?
And what set the primordial composition of baryons?
$\rightarrow$ events in the very early Universe...

Momentum Redshifting: Rigorously

the preceding heuristic arguments give the right result, but to obtain this rigorously requires General Relativity (if you haven't had GR yet, never mind)

in GR: a free particle's motion is a geodesic
so 4-momentum $p^\mu = m dx^\mu/ds = m(\gamma, \gamma\vec{v}) = (E, \vec{p})$ changes as

$$p^\alpha \nabla_\alpha p^\mu = p^\alpha \partial_\alpha p^\mu + \Gamma^\mu_{\alpha\beta} p^\alpha p^\beta = 0 \quad (19)$$

and we see that the change in u is due to the connection term Γ, i.e., to curvature
$\rightarrow$ curvature tells matter how to move

note: homogeneity hugely simplifies: $p^\mu = p^\mu(t)$
so $\partial_\mu p = 0$ except for $\partial_t p = \dot{p}$

consider the $\mu = i \in (x, y, z)$ component of the geodesic eq

$$p^\alpha \partial_\alpha p^i + \Gamma^i_{\alpha\beta} p^\alpha p^\beta = E\dot{p} + \Gamma^i_{\alpha\beta} p^\alpha p^\beta \quad (20)$$

$$= 0 \quad (21)$$

note that in FRW, if we write $ds^2 = dt^2 - h_{ij} dx^i dx^j$ where h_{ij} is the spatial metric, then nonzero $\Gamma^i_{\alpha\beta}$ are

$$\Gamma^i_{0j} = \frac{\dot{a}}{a} \delta^i_j \quad (22)$$

where δ^i_j is the Kronecker delta (try it!)

We then have

$$E\dot{p^i} + \frac{\dot{a}}{a} E p^i = 0 \quad (23)$$

and thus

$$d\vec{p}/dt = -\frac{\dot{a}}{a} \vec{p} \quad (24)$$

$$|\vec{p}| \propto \frac{1}{a} \quad (25)$$

Note that this result is completely general, i.e., works for all relativistic p, so

- in non-rel limit, $v \propto 1/a$: vel redshifts, and free particles eventually come to rest wrt the comoving background
- in ultra-rel limit, $v = p/E \approx c$, doesn't redshift, but since $E \approx p$, $E \propto 1/a$: energy redshifts

note classical derivation: didn't need Planck/de Broglie relation $p \propto 1/\lambda$ to show this (though that still works too)

Linear Theory II: Sketch of Relativistic Treatment

see, e.g., Dodelson text, Liddle & Lyth Ch. 14

Recall limits of Newtonian treatment:

- only appropriate for scales $\lambda \ll d_H$: sub-horizon
- relativistic effects like time dilation absent or *ad hoc*

General Relativistic approach to cosmological perturbations

- as in Newtonian analysis, perturb density, velocity
  $\rightarrow$ this perturbs stress-energy
  schematically "$\delta T \approx \delta\rho + \delta P = \delta\rho + c_s^2 \delta\rho$"
- must therefore add small perturbations to metric:
  $g_{\mu\nu} = g_{\mu\nu}^{\mathrm{FRW}} + h_{\mu\nu}$
- these are related by Einstein's Equation
  $G_{\mu\nu} \approx$ "$\partial\partial g^{\mathrm{FRW}} + \partial\partial h$" $= 8\pi G_N T_{\mu\nu} \approx$ "$8\pi G_N(\rho + \delta\rho)$"

Metric Perturbations

Perturbations to metric tensor can be classified as:

- *scalar* – density perturbations couple to these
  these are most important
- *vector* – velocity perturbations couple to these
  these are least important (perturbations decay with time)
- *tensor* – source of gravity waves
  inflationary quantum perturbation excite these modes!

focus on *scalar* perturbations, which modify FRW metric thusly:

$$(ds^2)_{\text{perturbed}} = a(\eta)^2 \left[(1+2\Psi)d\eta^2 - (1-2\Phi)\delta_{ij}dx^i dx^j\right] \quad (26)$$

Coordinate freedom $\leftrightarrow$ "gauge" choice $\leftrightarrow$ spacetime "slicing"

$\Rightarrow$ here: *"conformal Newtonian gauge"*:

- $\Psi(\vec{x},t), \Phi(\vec{x},t)$ Schwarzchild-like forms if $a=1, \dot{a}=0$

Substitute perturbed metric into Einstein, keep only linear terms in Φ and Ψ, e.g., neglect Φ^2

use conformal time

and go to k-space

- $\nabla_\mu T^{\mu 0} \rightarrow$ "continuity"

$$\frac{d\delta}{d\eta} + ikv + 3\frac{d\Phi}{d\eta} = 0 \tag{27}$$

- $\nabla_\mu T^{\mu i} \rightarrow$ "Euler"

$$\frac{dv}{d\eta} + \frac{da/d\eta}{a}v + ik\Psi = \text{pressure sources} \tag{28}$$

- $G_{\mu\nu} = 8\pi G_{\mathrm{N}} T_{\mu\nu} \rightarrow$ "Poisson"

$$k^2\Phi = -4\pi G a^2 \rho\delta \tag{29}$$

$$k^2(\Psi - \Phi) = -8\pi G a^2 \text{"}\langle P_x - P_y\rangle\text{"} \tag{30}$$

expect *anisotropic stress* small: $\langle P_x - P_y\rangle \ll \rho\delta \rightarrow$ $\boxed{\Psi \approx \Phi}$

Recall: conformal time η gives particle horizon

On *sub-horizon* scales $\lambda \sim 1/k \ll \eta$:
relativistic treatment gives back Newtonian result
in fact: justifies our Newtonian treatment

On *super-horizon* scales $\lambda \sim 1/k \gg \eta$:
relativistic treatment still valid
$\rightarrow$ will use this to follow inflationary perturbations
through horizon crossing