Astro 507
Lecture 41
May 5, 2014

Announcements:

- Final Preflight posted, due next Wednesday 9am
  fun, optional, easy bonus points
- **ICES** available online – please do it!

Final Problem Set (PS7): Due Thurs May 15

- takes place of final exam
- open book, notes, web
- but: do not collaborate!

Vote: how long do you want the PS?

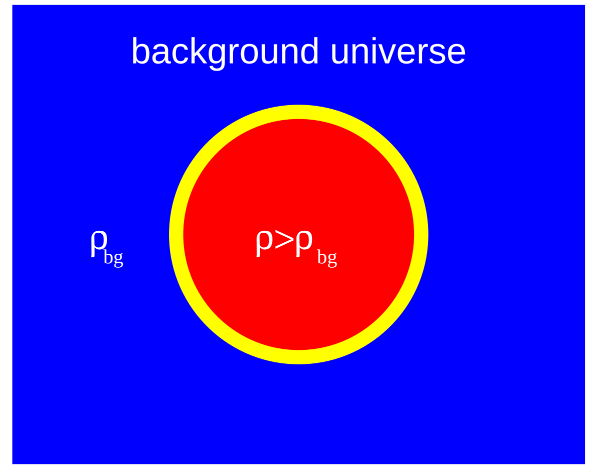
(a) 1 week
(b) 3 days
(c) 2 days
(d) 1 day

Last time: spherical collapse

idealized initial conditions

"top hat" Universe

- spherical, uniform density ρ
- embedded in flat, matter-dom universe with "background" density $\rho_{\rm bg}$ ("compensated" by surrounding underdense shell)



spherical collapse model a cosmological workhorse

a nonlinear problem with analytic solution!

Q: what is special/magical about this setup?

Q: qualitative results?

Q: why useful? limitations?

Spherical Collapse: Quantitative Lessons

want overdensity: since $\rho \propto 1/a^3$

$$\delta(t) = \frac{\rho(t)}{\rho_{\rm bg}(t)} - 1 = \left(\frac{a_{\rm bg}}{a}\right)^3 - 1 \qquad (1)$$

with $a_{\rm bg} \propto t^{2/3}$ the matter-dom background
→ exact nonlinear solution (pre-virial)

For small t, to first order $a(t) \sim t^{2/3} = a_{\rm bg}(t)$:
  background result; $\delta(t) = 0$
to second order: $a(t) = a_{\rm bg}(t)[1 - (12\pi t/t_{\rm coll})^{2/3}/20]$

$$\delta(t) \approx \frac{3}{20}\left(\frac{12\pi t}{t_{\rm coll}}\right)^{2/3} = \delta_{\rm lin}(t) \qquad (2)$$

$\delta(t) \propto D(t) \propto t^{2/3} \propto a_{\rm bg}$ same as linear result!

Very useful result:

$$\delta_{\text{nonlin}}(t) = \left(\frac{a_{\text{bg}}}{a_{\text{nonlin}}}\right)^3 - 1 \qquad (3)$$

$$\delta_{\text{lin}}(t) \approx \frac{3}{20}\left(\frac{12\pi t}{t_{\text{coll}}}\right)^{2/3} \qquad (4)$$

connects full nonlinear result with linear counterpart
$\rightarrow$ maps between the two

E.g., at turnaround
$\delta_{\text{nonlin}} = (6\pi)^2/4^3 = 5.6$, but $\delta_{\text{lin}} = 1.06$
at virialization (PS6):
$\delta_{\text{nonlin}} \approx 180$, but $\delta_{\text{lin}} = 1.69$
$\rightarrow$ defines a critical linear overdensity
Q: why useful?

Strategy: *given initial linear density field* δ_i

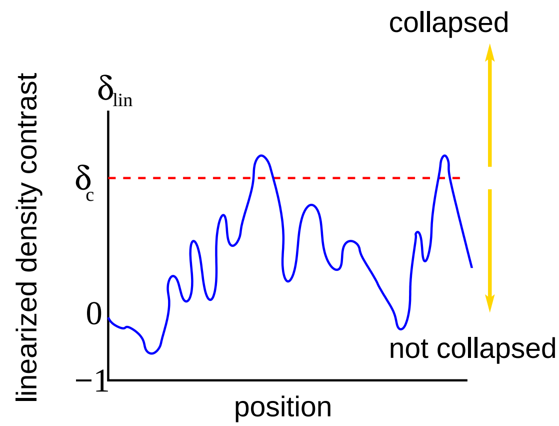
- evolve perturbations with linear growth $\delta_{\text{lin}}(t) = D(t)/\delta_i$
- identify linearly extrapolated perturbations with $\delta_{\text{lin}}(t) > 1.69$

  $\Rightarrow$ *these will be collapsed objects by time* t*!*

lesson: in *linearized* $\delta_{\text{lin}}(t_0)$

a "cut" at δ_c

divides virialized vs nonvirialized



also: in a *nonlinear field*, can use $\delta_{\text{vir}} \sim 180$

as working *definition* of collapsed structure

good for comparing theory, observation *Q: procedure?*

Nonlinear Evolution: Lessons from Spherical Collapse

Qualitatively

▷ overdensity evolves as closed "subuniverse"

▷ starts expanding, but slower than cosmic background
pulls away from Hubble flow: reach max expansion, then turnaround

▷ virialize → form bound object

▷ no further expansion, except due to accretion, merging

Quantitatively

▷ can compute both $\delta_{\rm lin}(t)$ and exact $\delta(t)$
gives mapping from *easy* to (more) *correct*

▷ collapse/virialization when $\delta_{\rm lin} = 1.69$ and $\delta = 18\pi^2 \simeq 180$
recipe for forecasting structures in initial field $\delta_{\rm init} \ll 1$
recipe for defining halos: region surrounding density peak
and having overdensity $\delta\rho/\rho \sim 180$

★ Given these, can devise analytical tools to describe distribution of structures

Press-Schechter Analysis

Outlook
adopt hierarchical picture (i.e., some form of CDM)
$\Rightarrow$ matter at *every* point belongs to some structure
over time: go from many small structures to fewer, larger ones

Goal
Given properties of density field–i.e., $P_{\mathrm{init}}(k)$ and $P(k,t) = T_k^2(t) P_{\mathrm{init}}(k)$
Compute distribution of structures as function of mass, time

Quantitatively: want "mass function"
comoving number density of structures
in mass range $(M, M + dM)$:

$$\frac{dn_{\mathrm{com}}}{dM}(M,t) \tag{5}$$

from this, can compute many other things
e.g., density in $(M, M + dM)$ *Q: which is...?*

Press-Schechter Ingredients/Assumptions

• given mass M, *filter* density field
on comoving length R such that $M = 4\pi/3\ \rho_{\mathrm{bg,com}}(t) R^3$
density contrast has *variance* $\sigma^2(M) = \int P(k)\ W(k;R)\ d^3k$

• *in linear regime,* density field obeys *Gaussian statistics*:
in filtered field, probability of finding contrast in $(\delta_{\mathrm{lin}}, \delta_{\mathrm{lin}} + d\delta_{\mathrm{lin}})$:

$$P(\delta_{\mathrm{lin}}; M, t)\ d\delta_{\mathrm{lin}} = \frac{1}{\sqrt{2\pi\sigma^2(M,t)}} \exp\left[-\frac{\delta_{\mathrm{lin}}^2}{2\sigma^2(M,t)}\right]\ d\delta_{\mathrm{lin}} \tag{6}$$

why only good in linear regime *Q: why?*

• Spherical collapse model maps from linear $\rightarrow$ nonlinear
identifies *linear contrast threshold* $\delta_c \simeq 1.69$ for collapsed objects
note: δ_c is time indep! (in EdS cosmo)
$\Rightarrow$ can find fraction of cosmic mass in objects of mass M
Q: how?

fraction of mass or of comoving volume
in collapsed objects of mass M at time t is

$$
\begin{aligned}
f(>\delta_c; M, t) &= \int_{\delta_c}^{\infty} P(\delta_{\rm lin}; M, t)\ d\delta_{\rm lin} && (7) \\
&= \frac{1}{\sqrt{2\pi\sigma^2(M,t)}} \int_{\delta_c}^{\infty} \exp\left[-\frac{\delta_{\rm lin}^2}{2\sigma^2(M,t)}\right]\ d\delta_{\rm lin} && (8) \\
&= \frac{1}{\sqrt{2\pi}} \int_{\delta_c/\sqrt{2}\sigma}^{\infty} e^{-u^2} \equiv \frac{1}{2}{\rm erfc}\left[\frac{\delta_c}{\sqrt{2}\sigma(M,t)}\right] && (9)
\end{aligned}
$$

- for realistic $P(k)$, $\sigma^2(M) \sim \int k^3 P(k) W_k(M) dk/k \sim M^{-(n+3)/3}$
  $\rightarrow$ at fixed mass, $\sigma(M,t)$ monotonically decreases with M
  (down to some minimum M cutoff)
- $\sigma(M,t)$ evolves (linearly) as $\sigma \sim a(t) \sim 1/(1+z)$

Q: implications for mass distribution at fixed time?
Q: implications for structure formation over time?

Press-Schechter: mass fraction and structure formation

$$f(>\delta_c; M, t) = \frac{1}{\sqrt{2\pi}} \int_{\delta_c/\sqrt{2}\sigma}^{\infty} e^{-u^2} = \frac{1}{2}\mathrm{erfc}\left[\frac{\delta_c}{\sqrt{2}\sigma(M,t)}\right] \tag{10}$$

⋆ mass distribution at fixed t:

as filter mass M decreases, variance $\sigma(M)$ increases

▷ more large fluctuations → more above threshold

▷ more structures at smaller masses

i.e., $\delta_c/\sqrt{2}\sigma(M)$ decreases → larger f

⇒ smallest halos most numerous → hierarchy of masses!

⋆ time evolution at fixed M:

at time, scale factor increases, variance $\sigma(t) \propto a(t)$ increases

▷ more structures at fixed mass

▷ small structures merge → larger (at expense of smallest)

⇒ hierarchical clustering!

Press-Schechter Mass Function I: Quick-n-Dirty

Press & Schechter (1974):
note that structures can only be made from *over*densities
but *under*densities (voids) occupy mass fraction $f(\delta_{\rm lin} < 0) = 1/2$
so fraction of *overdensites* in collapsed objects of M is

$$F(> \delta_c; M, t) = \frac{f(\delta_{\rm lin} > \delta_c)}{f(\delta_{\rm lin} > 0)} = 2 f(\delta_{\rm lin} > \delta_c) \tag{11}$$

famous factor of two!

Compare mass fraction at M and $M + dM$: difference

$$dF = F(M + dM) - F(M) \simeq \frac{dF}{dM}\, dM \tag{12}$$

$$= \sqrt{\left(\frac{2}{\pi}\right)} \frac{d\sigma(M)^{-1}}{dM} \frac{\delta_c}{\sigma(M)}\, e^{-\delta_c^2/2\sigma^2(M)}\, dM \tag{13}$$

But probability of finding structure M in filter volume $V_{\text{com}} = M/\rho_{\text{bg}}$ is

$$dF(M) = V \frac{dn}{dM}\, dM = \frac{M}{\rho_{\text{bg}}} \frac{dn}{dM}\, dM \qquad (14)$$

and so *PS mass function* is

$$M\frac{dn}{dM} = \frac{\rho_{\text{bg}}}{M}\, M\frac{dF}{dM} = \sqrt{\frac{2}{\pi}}\, \frac{d\ln\sigma(M)^{-1}}{d\ln M}\, \frac{\delta_c}{\sigma(M)}\, \frac{\rho_{\text{bg}}}{M}\, e^{-\delta_c^2/2\sigma^2(M)}$$

- implicitly also a function of t via $\rho_{\text{bg}}(t)$ and $\sigma(M,t)$
- encodes and quantifies hierarchical clustering

from this can immediately find, e.g., distribution of (comoving) density across masses of collapsed objects:

$$\frac{d\rho(M)}{dM} = M\frac{dn}{dM} \qquad (15)$$

Press-Schechter: Summary

Quantitative Output

⋆ Easy to use, very powerful (semi-)analytic mass function

Qualitative Worldview/Limitations

⋆ *every* point lies in exactly one structure:
largest above threshold

⋆ all structures have $\delta_{\rm lin} = \delta_c$: born today!

⋆ PS blind to interior substructure
and formation history of a given object

Q: how to test PS theory?

Q: which structures should be best described? worst?

Tests of Press-Schechter

Versus Numerical Simulations

PS is idealized analytic approximation of hierarchical clustering
assumes true density field δ perfectly mapped onto
linear field δ_{lin} vis spherical collapse model

Even if underlying CDM, hierarchy idea right, PS approximate
$\rightarrow$ test against numerical simulations w/ non-ideal δ field
results: unreasonably good agreement!

Versus Observations

Best applicable to those just formed: $\sigma(R) \sim \sigma_8 \sim 1$
$\rightarrow$ galaxy clusters! $M \sim 10^{15}\ M_\odot$, and so PS gives

$$n(M) \sim M \frac{dn}{dM} \sim \frac{\rho_0}{M} \nu e^{-\nu^2/2} \sim \frac{\rho_0}{M} \sim 10^{-4}\ \text{Mpc}^{-3} \tag{16}$$

about right! (where $\nu = \delta_c/\sqrt{2}\sigma \sim 1$)
...and works unreasonably well at other scales too

Applications of Press-Schechter

Mergers

PS very powerful because gives mass function vs time:

$$\mathcal{N}(M,t) = M\frac{dn}{dM}(t) \sim \nu(t)\ e^{-\nu^2(t)/2} \tag{17}$$

with

$$\nu(t) = \frac{\delta_c}{\sigma(M,t)} = \frac{\delta_c}{D(t)\sigma_{\rm init}(M)} = \frac{a(t_{\rm init})}{a(t)}\nu_{\rm init} \tag{18}$$

recall: $\sigma_{\rm init}(M)$ *decreases* with M *Q: why?*

So to find time change: just take derivative

$$\dot{\mathcal{N}} \sim |\dot{\nu}|(\nu^2 - 1)e^{-\nu^2/2} \sim \text{creation} - \text{destruction} \tag{19}$$

Q: merging for large, small ν? large, small M?

at fixed time t

$$\dot{\mathcal{N}} \sim |\dot{\nu}|(\nu^2 - 1)e^{-\nu^2/2} \tag{20}$$

small M $\rightarrow$ largest σ: $\nu = \delta_c/\sigma(m) < 1$

$\dot{\mathcal{N}} > 0$: net destruction

and so large M $\rightarrow$ net creation – at expense of small objects

PS Application II: Quasar Abundance

- Quasars must be massive (Eddington limit) black holes at galaxy centers $\rightarrow$ demands $M_{\rm gal} > M_{\rm bh} \gtrsim 10^{12} M_\odot$
- Quasars found out to high redshift $z > 3$ (in fact $\gtrsim 7$)

PS: can find number density of objects with $M > 10^{12} M_\odot$ at epoch $z = 3$

$$n_{\rm com}(> 10^{12} M_\odot; z = 3) = \int_{10^{12} M_\odot} \frac{dn}{dM} dM \sim 10^{-8} \ {\rm Mpc}^{-3} \tag{21}$$

about right!

Recombination Re-Revisited

so far: theory of small-scale CMB anisotropies worked in k space
- before recombination: modes are standing waves
- CMB records phase at recombination

but can also work in *real space*
- consider a single localized *overdensity*
- initially *adiabatic*

$$\delta_{\rm m}(t_{\rm init}) = \delta_{\rm b}(t_{\rm init}) = \delta_\gamma(t_{\rm init}) = \delta_\nu(t_{\rm init}) \qquad (22)$$

Q: pre-recombination initial behavior of the dark matter? baryons & photons? neutrinos?

Q: interesting scale(s)?

Before Recombination/Decoupling `www: simulations`

dark matter: cold, pressureless
overdensity grows with time, drawing in surrounding matter

baryon/photon fluid: high-pressure
fluid sees large *pressure gradient*: drives forces that try to smooth

- overdense, pressurized region propagates out at speed c_s
- generates a shell of comoving radius $r_{\rm com} \sim c_s \eta$
- shell continues until recombination, when radius is

$$r_{\rm shell,com} = \int c_s d\eta \approx c_s \eta_{\rm dec} \sim 150 \text{ Mpc} \tag{23}$$

neutrinos: hot, pressureless
fly out at speed c from overdensity
continue until nonrelativistic

Q: post-recombination/decoupling behaviors?
Q: effect of DM on baryon/photon fluid? on neutrinos?

At decoupling: baryonic "rings" at $r_{\rm shell,com} \approx c_s \eta_{\rm dec} \sim 150$ Mpc

After Recombination/Decoupling `www: simulations`

baryon/photon fluid: attracted by central DM potential

- nearby baryons falls in
- distant ring feature remains

dark matter: attracted by baryonic feature at $r_{\rm shell,com}$

- DM also forms rings at $r_{\rm shell,com}$
- overdensity lower than center by $\sim \Omega_{\rm b}/\Omega_{\rm m} \sim 1/7$

neutrinos: attracted to overdensities
but while relativistic, smooth perturbations

Q: what if many local perturbations? observable signature?

Baryon Acoustic Oscillations

around recombination, perturbations still linear

- density field well-described by superposition
- overdensities all surrounded by rings at $r_{\text{shell,com}}$
- randomness of initial field obscures ring patterns
- but still excesses of mater 150 Mpc away from other excesses
  $\Rightarrow$ *correlations are observable!*

in real space: correlation function

$$\xi(r) = \langle \delta(\vec{x}) \; \delta(\vec{x} + \vec{r}) \rangle \tag{24}$$

Q: what should we see?

`www: SDSS data`

in k space: power spectrum

sharp feature in real-space $\rightarrow$ oscillations in $P(k)$

Q: why is this incredibly powerful?

Director's Cut Extras

Press-Schechter II: Excursion Sets

More sophisticated (and insightful) derivation of same result

Sketch of procedure:
1. given initial density field and (Gaussian) filter window
2. pick a point $\vec{x}$ in space, filter over neighborhood R, mass $M(R)$
3. scan down in mass: at $M \to \infty$, $\sigma(M) \to 0$ *Q: why?*
   and so filtered $\delta(\vec{x})_M = 0$
3. as M decreases, $\sigma(M)$ increases
   filtered $\delta(\vec{x})_M \neq 0$, alternates sign, amplitude
   $\Rightarrow$ $\delta(\vec{x})_M$ is a random walk vs $\sigma(M)$! exactly!
4. can ask: at which M does $\delta(\vec{x})_M$ first cross threshold δ_c
   $\Rightarrow$ this sets M of structure containing point $\vec{x}$
5. repeat for all $\vec{x}$ and average $\to$ PS distribution follows!

Q: limitations/implicit assumptions?