Astro 507
Lecture 5
Jan 31, 2014

Announcements:

- **Happy New Year!** have a treat
- PS1 due next Friday, Feb. 7
  Director's Cut Extras today: magnitude scale
- Office Hours: 3:10–4:00 pm Thursday, or by appointment
  note phase correlation with Friday due date
- Preflight 1 was due today–thanks!

Last time: an expanding universe

- *Q: how do we describe cosmic kinematics = particle motions?*
- *Q: what is $a(t)$ physically? units? values?*
- *Q: why is a important cosmologically?*
- *Q: what is a "comoving" coordinate?*
- *Q: how should cosmic matter density ρ depend on a?*

Density Evolution: Matter

definition: to cosmologist
matter $\equiv$ *non-relativistic* matter

in the non-relativistic regime:

- particle speeds $v \ll c$,
  and/or $kT \ll mc^2$ (particle rest energy)
- mass is *conserved*

in comoving sphere with volume $V \propto a^3$, mass conservation gives:

$$dM = d(\rho V) \propto d(\rho a^3) = 0 \tag{1}$$

gives density

$$\rho_{\text{non-rel}} \propto \frac{1}{V} \propto a^{-3} \tag{2}$$

density scaling with a:

$$\rho_{\rm non-rel} \propto \frac{1}{V} \propto a^{-3} \tag{3}$$

today: $\rho_{\rm matter}(t_0) \equiv \rho_{\rm m,0}$

so at other epochs (while still non-relativistic):

$$\boxed{\rho_{\rm m} = \rho_{\rm m,0}\ a^{-3}} \tag{4}$$

Q: what is $\dot{\rho}_{\rm m}$?

Matter Density: Time Change

matter density depends only on scale factor:

$$\rho_{\mathrm{m}} = \rho_{\mathrm{m},0} \; a^{-3} \tag{5}$$

and so

$$\dot{\rho_{\mathrm{m}}} = -3 \; \rho_{\mathrm{m},0} \; \dot{a} \; a^{-4} = -3H\rho_{\mathrm{m}} \tag{6}$$

Hubble sets rate for density decrease!

Q: how must this be altered in the steady-state cosmology?

Matter and the Steady State Cosmology

steady-state cosmology adopts *perfect cosmological principle*:
▷ homogeneous + isotropic + time invariant
a *non-evolving universe*

this demands $\dot{\rho} = 0$: *density constant*
but expansion carries galaxies away!
→ must be new matter created to replace it
mass creation rate per unit volume: q:

$$\frac{d(\rho a^3)}{dt} = q\ a^3 \qquad (7)$$

$$\dot{\rho} + 3H\ \rho = q \qquad (8)$$

to maintain steady state: creation rate density must be

$$\begin{aligned} q &= 3H\rho \\ &\approx 6 \times 10^{-47}\ \mathrm{g\ cm^{-3}\ s^{-1}} = 10^{-6}\ \mathrm{GeV}/c^2\ \mathrm{cm^{-3}\ Gyr^{-1}} \end{aligned}$$

Q: implications?

Alternative Derivation: Fluid Picture

in fluid picture: mass conservation $\rightarrow$ continuity equation

$$\partial\rho/\partial t + \nabla \cdot (\rho \vec{v}) = 0 \quad (9)$$

put $\rho = \rho(t)$ and $\vec{v} = H\vec{r}$:

$$\dot{\rho} + H\rho\nabla \cdot \vec{r} = \dot{\rho} + 3\frac{\dot{a}}{a} \quad (10)$$

$$\frac{d\rho}{\rho} = -3\frac{da}{a}\rho \quad (11)$$

$$\rho \propto a^{-3} \quad (12)$$

Cosmodynamics Computed

cosmic dynamics is evolution of a system which is

- *gravitating*,
- *homogeneous*, and
- *isotropic*

Complete, correct treatment: General Relativity
$\Rightarrow$ we will sketch this starting next week

quick 'n dirty:
Non-relativistic (Newtonian) cosmology
pro: gives intuition, and right answer
con: involves some ad hoc assumptions only justified by GR

Inputs:

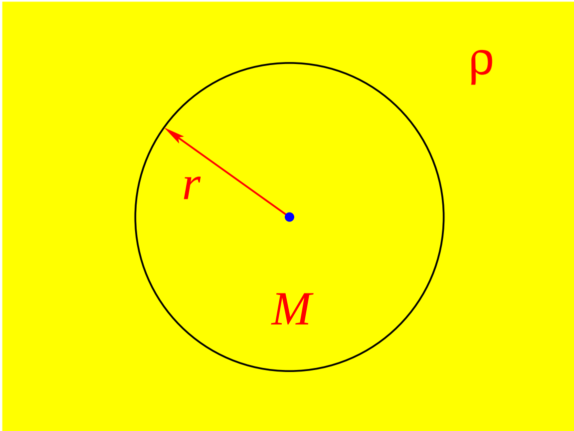
- arbitrary cosmic time t
- cosmic mass density $\rho(t)$, spatially uniform

- cosmic pressure $P(t)$: in general, comes with matter
  but for non-relativistic matter, P not important source of
  energy and thus mass ($E = mc^2$) and thus gravity
  so ignore: take $P = 0$ for now (really: $P \ll \rho c^2$)

thus: *gravity is only force*
all cosmic matter is in "*free fall*"

Construction:

pick arbitrary point $\vec{r}_{\rm center} = 0$,
surround by comoving sphere, radius $r(t)$
that moves in order to always enclose
some arbitrary but fixed mass



$$M(r) = \frac{4\pi}{3} r^3 \rho = const \qquad (13)$$

consider a point on the sphere

Q: is it accelerated?

Q: what is $\ddot{\vec{r}} = ?$

Newtonian Cosmodynamics

a point on the sphere feels acceleration

$$\ddot{\vec{r}} = \vec{g} = -\frac{GM}{r^2}\hat{r} \tag{14}$$

with pressure $P = 0$

multiply by $\dot{\vec{r}}$ and integrate:

$$\dot{\vec{r}} \cdot \frac{d}{dt}\dot{\vec{r}} = -GM\frac{\hat{r} \cdot d\vec{r}/dt}{r^2} \tag{15}$$

$$\frac{1}{2}\dot{r}^2 = \frac{GM}{r} + K = \frac{4\pi}{3}G\rho r^2 + K \tag{16}$$

Q: physical significance of K? of it's sign?

Q: what happens when we introduce scale factor?

Friedmann (Energy) Equation

introduce cosmic scale factor: $r(t) = a(t)\ r_0$

"energy" eqn: **Friedmann equation**

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho - \frac{\kappa c^2}{R^2 a^2} \tag{17}$$

we will see: full GR gives $K = -2r_0^2(\kappa c^2/R^2)$
where

- $\kappa = \pm 1, 0$, and
- const R is lengthscale: "*curvature*" of U.

In full GR:

▷ Friedmann eq. holds even for relativistic matter, but

▷ where $\rho = \sum_{\text{species},i} \varepsilon_i/c^2$: mass-energy density

The Mighty Friedmann (Energy) Equation

fundamental equation of the Standard Cosmology:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho - \frac{\kappa c^2}{R^2 a^2} \tag{18}$$

Q: why is it so important?

Q: what's a variable?

Q: what's a parameter?

Q: $a(t)$ *behavior if* $K = \kappa = 0$*? if* $\kappa \neq 0$*?*

Dissecting Friedmann

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G\rho - \frac{\kappa c^2}{R^2 a^2} \qquad (19)$$

variables change with time

- a: cosmic scale factor
- ρ: total cosmic mass-energy density

parameters constant, fixed for all time

- $\kappa = \pm 1$ or 0: sign of "energy" (curvature) term
- R: characteristic lengthscale, GR $\rightarrow$ curvature scale

Q: how does expansion of U depend on contents of U?

Q: how does expansion of U not *depend on contents of U?*

Q: what about acceleration–$\ddot{a}$?

Friedmann Acceleration Equation

Newtonian analysis gives $\ddot{a}$ for $P \to 0$
In full GR: with $P \neq 0$, get Friedmann **acceleration** eq.

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3} G(\rho + 3P/c^2) \tag{20}$$

Pressure and Friedmann

⋆ in "energy" ($\dot{a}$) eq.: P *absent*, even in full GR
⋆ in acceleration eq., GR $\to$ P present, *same* sign as ρ
adds to "active gravitational mass"
Q: why? Q: contrast with hydrostatic equilibrium?

Friedmann energy eq is "equation of motion" for scale factor
i.e., governs evolution of $a(t)$.
To solve, need to know how ρ depends on a
Q: how figure this out?

A Matter-Only Universe

consider a universe containing *only* non-relativistic matter
Friedmann:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{\kappa c^2}{R^2}\frac{1}{a^2} \tag{21}$$

$$= \frac{8\pi G}{3}\rho_0 a^{-3} - \frac{\kappa c^2}{R^2}a^{-2} \tag{22}$$

For $\kappa = 0$: "Einstein-de Sitter"

$$(\dot{a}/a)^2 = \frac{8\pi G}{3}\rho_0 a^{-3} \tag{23}$$

evaluate today: $H_0^2 = 8\pi G\rho_0/3$

$$a^{1/2}da = H_0\,dt \tag{24}$$

$$2/3\ a^{3/2} = H_0\,t \tag{25}$$

Q: implicit assumptions in solution?

Einstein-de Sitter:

$$t = \frac{2}{3} a^{3/2} H_0^{-1} \tag{26}$$

$$a = \left(\frac{3}{2} H_0 t\right)^{2/3} = \left(\frac{t}{t_0}\right)^{2/3} \tag{27}$$

Now unpack the physics:

- boundary condition: $a = 0$ at $t = 0 \rightarrow$ "big bang"
- $a \propto t^{2/3}$ *Q: interpretation?*
- evaluate Hubble parameter

$$H = \frac{\dot{a}}{a} = \frac{2}{3}\frac{1}{t} \tag{28}$$

Q: interpretation?

- present age:

$$t_0 = \frac{2}{3} H_0^{-1} = \frac{2}{3} t_{\rm H} \tag{29}$$

Hubble time $t_{\rm H}$ sets scale

Q: note that $t_0 < t_{\rm H}$: why?

Other Einstein-de Sitter fun facts:

- U. half its present age at $a = 2^{-2/3} = 0.63$
- objects half present separation (and $8\times$ more compressed) at $t = 2^{-3/2} t_0 = 0.35 t_0$
- using measured value of H_0, calculate $t_0 = 8.9$ Gyr but know globular clusters have ages $t_{gc} \gtrsim 12$ Gyr *Q: huh?*

Director's Cut Extras: The Magnitude Scale

Star Brightness: Magnitudes

star brightness (flux) measured in **magnitude** scale
magnitude = "rank" : smaller $m \rightarrow$ **brighter**, *more* flux
Sorry.

Magnitudes use a logarithmic scale:

- difference of 5 mag is factor of 100 in flux:

$m_2 - m_1 = -2.5 \log_{10} F_2/F_1$ (definition of mag scale!)

- mag units: dimensionless! (but usually say "mag") since always a log of ratio of two dimensionful fluxes with physical units like W/m^2

What is mag difference $m_2 - m_1$:
Q: if $F_2 = F_1$?
Q: what is sign of difference if $F_2 > F_1$?
Q: for equidistant light bulbs, $L_1 = 100$Watt, $L_2 = 50$Watt?

Apparent Magnitude

a measure of star flux = (apparent) brightness

- no distance needed
- arbitrary mag zero point set for convenience:
  historically: use bright star Vega: $m(\text{Vega}) \equiv 0$
  then all other mags fixed by ratio to Vega flux

- ex: Sun has **apparent** magnitude $m_{\odot} = -26.74$
  i.e., $-2.5 \log_{10}(F_{\odot}/F_{\text{Vega}}) = -26.74$
  so $F_{\text{Vega}} = 10^{-26.74/2.5} F_{\odot} = 2 \times 10^{-11} F_{\odot}$
- ex: Sirius has $m_{\text{Sirius}} = -1.45 \rightarrow$ brighter than Vega

so: $F_{\text{Sirius}} = 3.8 F_{\text{Vega}} = 8 \times 10^{-11} F_{\odot}$

- ex: $m_{\text{Polaris}} = 2.02$ *Q: rank Polaris, Sirius, Vega?*

⋆ if distance to a star is known
can also compute Absolute Magnitude

abs mag M = apparent mag if star placed at $d_0 = 10$ pc

Q: what does this measure, effectively?

Absolute Magnitude

absolute magnitude M = apparent mag at $d_0 = 10$ pc

places all stars at constant fixed distance
$\rightarrow$ a stellar "police lineup"
$\rightarrow$ then differences in F only due to diff in L
$\rightarrow$ absolute mag effectively measure luminosity

Sun: abs mag $M_\odot = 4.76$ mag
Sirius: $M_{\rm Sirius} = +1.43$ mag
Vega: $M_{\rm Vega} = +0.58$ mag
Polaris: $M_{\rm Polaris} = -3.58$ mag
ϵ Eridani: $M_{\epsilon{\rm Eri}} = +6.19$ mag (nearest exoplanet host; $d = 3.2$ pc)
Q: rank them in order of descending L?

Immediately see that Sun neither most nor least luminous star around

Distance Modulus

take ratio of actual star flux vs "lineup" flux
at abs mag distance $d_0 = 10$ pc:

$$\frac{F}{F_0} = \frac{L/4\pi d^2}{L/4\pi d_0^2} \tag{30}$$

which, after simplification, leads to

$$m - M = 5 \log \left(\frac{d}{10 \text{ pc}} \right) \tag{31}$$

- depends only on distance d, not on luminosity!
  can use as measure of distance
- $m - M \equiv$ "distance modulus", sometimes denoted μ