Astro 507

Lecture 8

Feb. 7, 2014

Announcements:

- **PS1 due now**

- *Preflight 2 posted, due next Friday*

Last time:

- observations, theory, good taste all say: $\Omega_0 = 1$!

- but we also measure $\Omega_{\rm matter} \approx 0.3$!?

- end of the road for Newtonian Cosmology:
  this situation requires General Relativity!

Q: why?

Beyond Newton

Thus far: Newtonian cosmology
- develops intuition
- correct over small scales $\ll d_H$

Shortcomings:
- some features "pulled of out a hat"
  e.g., curvature scale R
  presence, coefficient of pressure
- Newtonian physics is incomplete (=wrong!)
$\Rightarrow$ *the Universe is relativistic!*

General Relativity

Relativity for the Impatient Cosmologist

For *General Relativity newcomers*, we will:
- sketch how GR generalizes special relativity
- sketch basic concepts of GR
- qualitatively discuss similarities, differences
  with special relativity, Newtonian Gravity
- No substitute for a real, rigorous, in-depth course:
  take General Relativity!

For *General Relativity veterans*, we will:
- sketch how Einstein equations → Friedmann eqs

For *everyone*, we will:
- show how the Cosmological Principle
  strongly constrains possible cosmic spacetimes
- semi-derive the cosmic (FLRW) metric
- use this to probe lifestyles in an expanding universe

Spacetime

evolving view of space, time, and motion:

Aristotle → Galileo → Einstein

Key basic concept: **event**

occurrence localized in space and time

e.g., firecracker, finger snap

idealized → no spatial extent, no duration in time

a goal (*the* goal?) of physics:

describe relationships among events

Q: consider collection of all possible events–what's included?

Spacetime Coordinates

Each event specifies a unique point in
spacetime = collection of all events

lay down coordinate system: 3 space coords, one time
4-dimensional, but as yet time & space always "orthogonal"

example:
a time t and Cartesian (x, y, z)
specify event $\rightarrow$ (t, x, y, z)

physics asks (and answers) what is the relationship
between two events, e.g., (t_1, x_1, y_1, z_1) and (t_2, x_2, y_2, z_2)

Represent spacetime geometrically: *spacetime diagram*
e.g., plot (x, t) coordinate plane
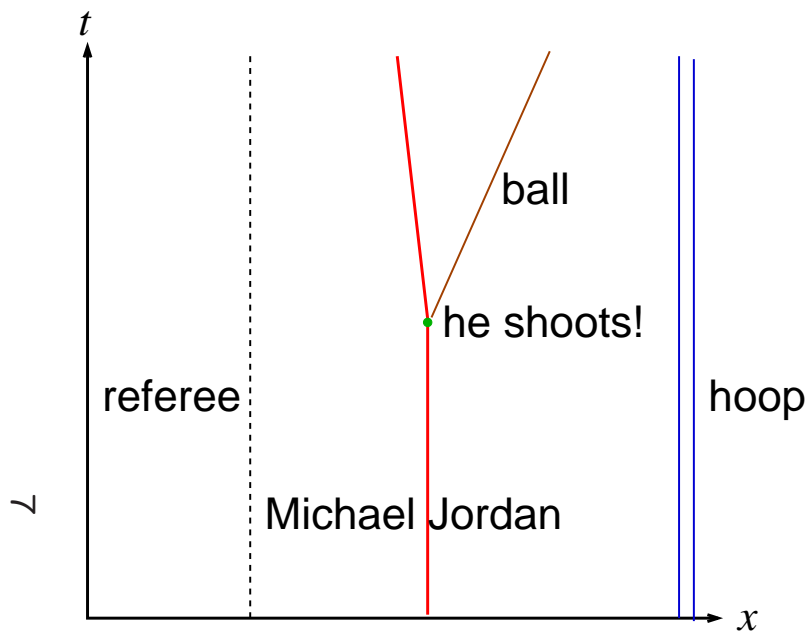*Q: one event? one observer at rest? a jump shot?*

Spacetime Diagram

objects (observers) at *rest*:
same x, y, z always, t ticks forward
geometrically, a line in spacetime: **"world line"**
if at rest: world line vertical

constant speed: $x = vt$: diagonal line

Galilean Relativity

consider two identical laboratories
(same apparatus, scientists, funding, vending machines)
move at constant velocity wrt each other

Galileo:
no experiment either can do (without looking outside)
will answer "which lab is moving"
→ *no absolute motion*, only relative velocity

Newton: laws of mechanics invariant
for observers moving at const v
"inertial observers"

Implications for spacetime
no absolute motion → *no absolute space*
(but still no reason to abandon absolute time)

Trouble for Galileo

Maxwell: equations govern light
very successful, but:
- predicts unique (constant) light speed c—relative to whom?
- Maxwell eqs **not** Galilean invariant

Lorentz: Maxwell eqs invariant when

$$
\begin{aligned}
t' &= \gamma(t - vx/c^2) & (1)\\
x' &= \gamma(x - vt) & (2)\\
y' &= y & (3)\\
z' &= z & (4)
\end{aligned}
$$

where $\gamma = 1/\sqrt{1 - v^2/c^2} \geq 1$

Einstein:
Lorentz transformation not just a trick
but correct relationship between inertial frames!
$\Rightarrow$ this is the way the world is

Einstein: Special Relativity

consider two events

(t, x, y, z) and $(t + \Delta t, x + \Delta x, y + \Delta y, z + \Delta z)$

different inertial observers *disagree* about

i.e., measure different values for: Δt *and* $\Delta \vec{r}$

but all *agree* on $=$ calculate same value of

the **interval**

$$\Delta s^2 \equiv (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 \qquad (5)$$
$$= (c\Delta t)^2 - (\Delta \ell)^2 \qquad (6)$$

everyone agrees on value $=$ *Lorentz invariant*

Note: interval can have $\Delta s^2 > 0, < 0, = 0$

Light pulse:

in rest frame of flash: photon positions $\Delta \ell = c\Delta t$

calculate interval: $\Delta s_{\mathsf{light}} = 0$
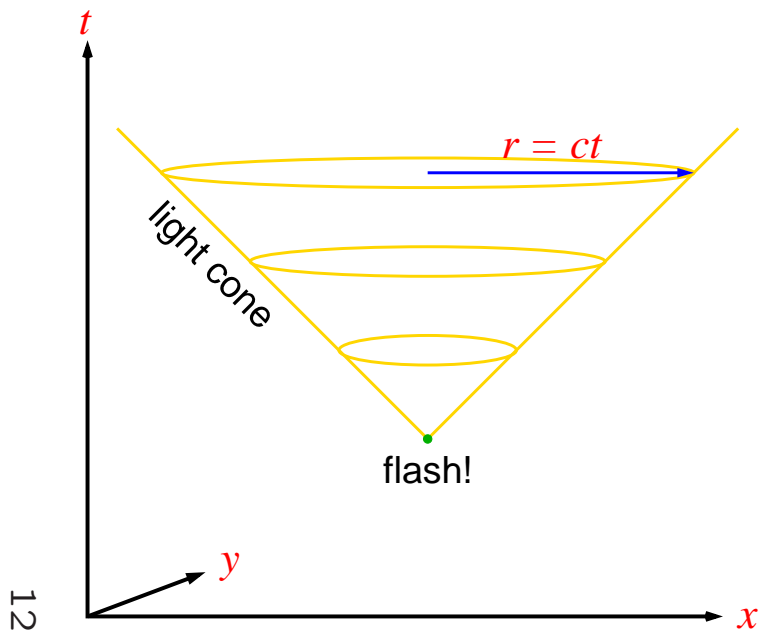
$\rightarrow$ *light moves at c in all frames!*

  all observers measure same speed of light!

Q: light flash in spacetime diag?

Light Pulse in Spacetime

in spacetime: light pulse at origin $(t, x, y, z) = (0, 0, 0, 0)$
moves so that distance $r = \sqrt{x^2 + y^2 + z^2} = ct$
geometrically: **light cone**

Motion and time:

Consider *two events, at rest in one frame:*

$\Delta \vec{x}_{\mathsf{rest}} = 0$ in rest frame, so

$\Delta s = c \Delta t_{\mathsf{rest}}$: $c \times$ elapsed time in rest frame

In another inertial frame, relative speed v:

events separated in space by $\Delta x' = v \Delta t'$

$$\Delta s = \sqrt{c^2 \Delta t'^2 - \Delta x'^2} = \sqrt{c^2 - v^2}\ \Delta t' = \frac{1}{\gamma} c \Delta t' \qquad (7)$$

since Δs same: infer $\Delta t' = \gamma \Delta t_{\mathsf{rest}} > \Delta t_{\mathsf{rest}}$

$\Rightarrow$ moving clocks appear to run slow

   (special) relativistic time dilation

$\Rightarrow$ no absolute time (and no absolute space)

Note: more on Special Relativity in Director's Cut Extras to today's notes

H. Minkowski:

"Henceforth, space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality."

The Speed of Massive Particles

Special relativity general rule: $v = p/E$
where E is total energy (see Extras to notes)
good for particles of any mass $m \geq 0$

...and where we have and will set $c = 1$

you can show that with explicit c factors, $v/c = cp/E$

but E and p also connected via
invariant $E^2 - p^2 = m^2$

$$v = \frac{\sqrt{E^2 - m^2}}{E} = \sqrt{1 - \left(\frac{m}{E}\right)^2} \tag{8}$$

Q: implications? what if $m = 0$? $m \neq 0$?

Causality and Spacetime

any particle of total energy E, mass m

moves at speed $v(E) = \sqrt{1 - \left(\frac{m}{E}\right)^2}$
- massive particles have $0 \le v < c$
- massless particles (e.g., γ) have $v = c$

$\Rightarrow v = c = 1$ *is universal speed limit*

$\Rightarrow$ *cannot transmit particles, info any faster*

future light cone at spacetime point p
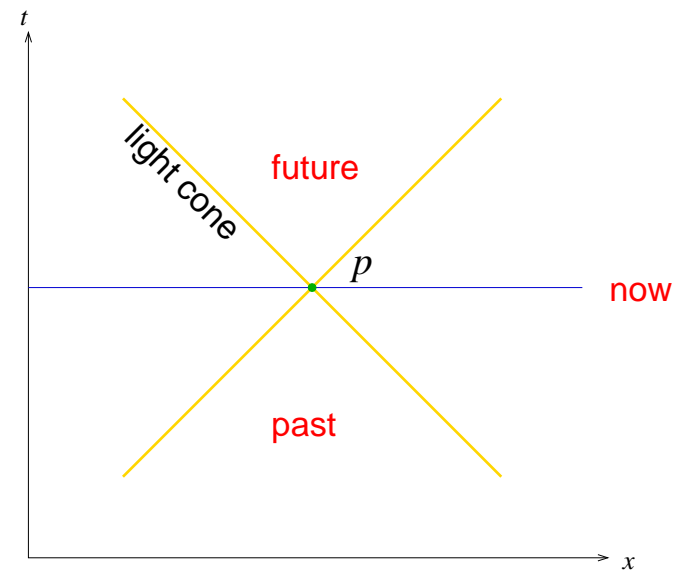
encloses region within which
  particles/info can move

i.e., *region p can influence*

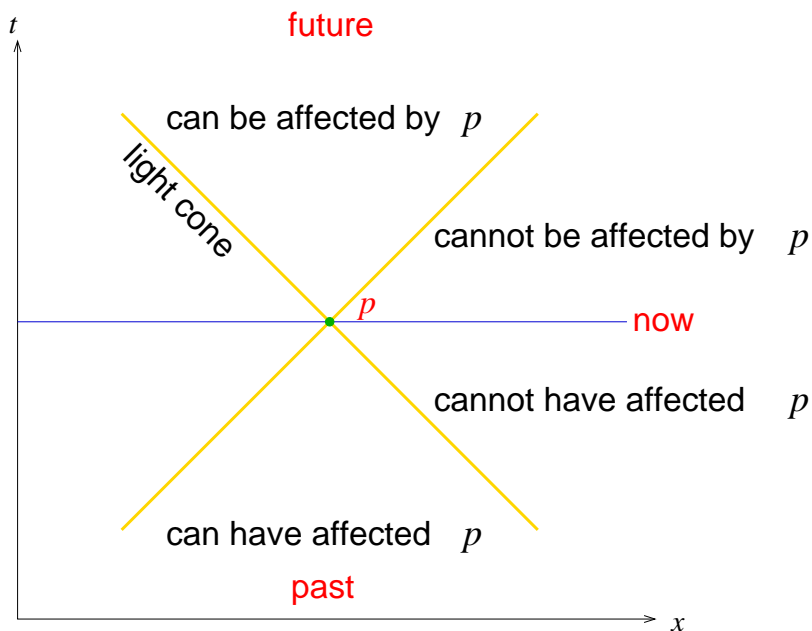$\Rightarrow$ future light cone is spacetime region
  causally connected to p

past light cone at p Q: ?

past light cone at p

 events in cone can send particles/info to p

 i.e., region which could have influenced p

 $\Rightarrow$ past light cone=causally connected to p

t

future

can be affected by p

light cone

cannot be affected by p

p

now

cannot have affected p

can have affected p

past

x

Q: two events causally connected if?

Q: sufficient or just necessary?

What About Gravity?

A. Einstein (1905):
   Newtonian dynamics $\rightarrow$ relativistic dynamics
   space, time $\rightarrow$ spacetime forever more

Relativity and classical fields:
- E&M: Maxwell eqs relativistically *OK*! (*motivated* Lorentz, SR)
- Newtonian gravity: $\vec{g} = -\nabla\phi = -Gm/r^2\,\hat{r}$ and $\nabla^2\phi = 4\pi G\rho$
  an *unmitigated disaster* *Q: Why?*

How to fix?
First attempt: analogy with electrostatics *Q: why?*

$$\nabla^2\phi - \partial_t^2\phi = 4\pi G\rho \tag{9}$$

- bad news: disagrees with expt (gives no light bending)
- good news: right "flavor"
  e.g., operator $\nabla^2 - \partial_t^2 \rightarrow$ waves $\rightarrow$ gravitational radiation!

Mystic Pisa

Experiment: Galileo (Tower of Pisa?)

free fall independent of mass, size, shape, composition

Q: lawyer's fine print?

Theory: Newton

always: $\vec{F} = m\vec{a}$

gravity: mass is "coupling strength" $\Rightarrow \vec{F}_{\text{grav}} = m\vec{g}$

$\Rightarrow$ free fall has $\vec{a} = \vec{g} \rightarrow$ indep of object properties

interesting curiosity

Theory: Einstein

gravity is acceleration, so maybe *acceleration is gravity*

i.e., their physical effects indistinguishable/equivalent

Equivalence Principle

T-shirt summary (R. Wald):
all bodies fall the same way in a gravitational field

an observer in free fall *Q: meaning?*
   cannot perform any experiment to determine whether
   she is in a gravitational field

an observer undergoing acceleration
   cannot perform any experiment to determine whether
   she is in a gravity field or an accelerating spacecraft

Equivalence Principle and Spacetime

Gedankenexperiment: accelerating spaceship
in open space (no massive objects nearby)
rockets blast to keep constant a

- Experiment 1: horizontal flashlight

Q: how will astronaut perceive beam path?
Q: which means? and upon applying equivalence principle...?

- Experiment 2: light signals between top & bottom

each flashes every $\tau_{em} = 1$ sec
according to emitter's clock
emission frequency $t_{em} = 1/\tau_{em}$

Q: what is change in top clocks' speed when pulse arrives?
Q: what frequency does top clock observe?

asymmetry: top clock accelerates away from bottom flash

→ relative speed changes during light transit

by amount $\delta v_{\text{top}} = -a\delta t \simeq -ah/c$

sign → receding from source

→ top observer sees freq Doppler shifted downward: *redshift*

$$f_{\text{obs,top}} \approx \left(1 - \frac{\delta v}{c}\right) f_{\text{em,bottom}} \qquad (10)$$

so top observer sees bottom flash period as

$$\frac{\tau_{\text{obs}} - \tau_{\text{em}}}{\tau_{\text{em}}} = \frac{\delta\tau}{\tau} = -\frac{\delta f}{f} \approx \frac{\delta v}{c} = +\frac{ah}{c^2} \qquad (11)$$

Q: which means? and upon applying equivalence principle...?

Equivalence Principle: in uniform gravity $g \to$ same results

- drooping flashlight beam: *gravity bends light!*

  `www:   strong lensing`

- shifted frequencies: *gravitational redshift/blueshift!*

- period shift: *gravitational time dilation*

$$\frac{\delta t}{t} = \frac{\delta \lambda}{\lambda} \approx \frac{gh}{c^2} = \frac{\phi}{c^2} \tag{12}$$

attic clocks faster than basement clocks: verified experimentally!

  `www:   Pound–Rebka expt`

  in weak gravity: shift $\approx \phi/c^2$

Note: gravity distorts

- light path (space)

- apparent frequency (time)

$\to$ gravity alters spacetime!

Einstein (1915): include gravity in spacetime

Director's Cut Extras: Special Relativity

Pre-Relativity: Aristotle

x, y, z Cartesian (Euclidean geometry)
spatial distance ℓ between events is:

$$\ell^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \qquad (13)$$

  and is independent of time
elapsed time between events is: $t_2 - t_1$
  and is independent of space
"absolute space" and "absolute time"

Is a particle at rest? $\Leftrightarrow$ do (x, y, z) change?
$\rightarrow$ "absolute rest, absolute motion"

Diagram: Aristotelian spacetime
unique "stacking" of "time slices"

Life According to Aristotle

Note: even in absolute space
event location indep of coordinate description
e.g., two observers choose coordinates different by a rotation:
(x, y) and $(x', y') = (x \cos \theta - y \sin \theta, y \cos \theta + x \sin \theta)$
preserves distance from origin: $x^2 + y^2 = (x')^2 + (y')^2$

objects (observers) at rest:
same x, y, z always, t ticks forward
geometrically, a line in spacetime: **"world line"**
if at rest: world line vertical
constant speed: $x = vt$: diagonal line

light: moves at "speed of light" c
$\rightarrow$ well-defined, since motion absolute
in spacetime: light pulse at origin $(t, x, y, z) = (0, 0, 0, 0)$
moves so that distance $\ell = \sqrt{x^2 + y^2 + z^2} = ct$
geometrically: **light cone**

Galilean Frames

each inertial obs has own personal frame:

obs ("Angelina") at rest in own frame: (x, y, z) same for all t

but to another obs ("Brad") in relative motion $\vec{v} = v\hat{x}$

B sees A's frame as time-dependent:

$$x_{\text{A as seen by B}} = x' = x - vt \tag{14}$$

but still absolute time: $t' = t$

Newton's laws (and Gravity) hold in both frames

can show: $d^2\vec{r}/dt^2 = \vec{F}(\vec{r}) \;\Rightarrow\; d^2\vec{r}'/dt'^2 = \vec{F}(\vec{r}')$

"Galilean invariance"

Geometrically:

different inertial frames $\rightarrow$ transformation of spacetime

slide the "space slices" at each time

(picture "shear," or beveling a deck of cards)

Spacetime and Relativity

Pre-Relativity: space and time separate and independent
but *rotations* mix *space* coords, e.g.,

$$x' = x\cos\theta - y\sin\theta \quad ; \quad y' = y\cos\theta + x\sin\theta \qquad (15)$$

without changing underlying vector (rotation of coords only)
transform rule holds not only for $\vec{x}$
but all other physical directed quantities: e.g., $\vec{v}, \vec{a}, \vec{p}, \vec{g}, \vec{E}$

Lesson: express & guarantee underlying rotational invariance
  by writing physical law in vector form
e.g., $\vec{F} = m\vec{a}$ gives same physics for any coord rotation

In special relativity:

spatial rotations still allowed, but also...

"boosts" from one frame to another
with relative speed $\vec{v} = v\hat{x}$

$$
\begin{align}
t' &= \gamma(t - vx/c^2) \tag{16} \\
x' &= \gamma(x - vt) \tag{17} \\
y' &= y \tag{18} \\
z' &= z \tag{19}
\end{align}
$$

- truly mix space and time $\rightarrow$ spacetime
- look like rotations, but 4-dimensional

$\rightarrow$ should express laws in terms of 4-D vectors:

"4-vectors," t, x components transform via Lorentz

Velocity, Momentum, Energy

Velocity:

for events separated by $dx^\mu = (dt, d\vec{x})$, put

$$u^\mu = \frac{dx^\mu}{ds} = (\frac{dt}{ds}, \frac{d\vec{x}}{ds})$$

(20)

covariant: written this way, a 4-vector:

transforms in boost a la Lorentz

i.e., *components are different* in different frames

but underlying physical entity frame-independent

"like with space vectors and rotations"

norm ("length") of 4-velocity

$$u \cdot u = \left(\frac{dt}{ds}\right)^2 - \left(\frac{d\vec{x}}{ds}\right)^2 = \frac{dt^2 - d\vec{x}^2}{ds^2} = \frac{ds^2}{ds^2} = 1$$

same number for all observers: *invariant*

Now want 4-momentum p^μ:

consider particle of (rest) mass m

where: rel. p^i should go to $m\vec{v}$ for small v

try: $p^\mu = mcu^\mu$

space part: $\boxed{\vec{p} = \gamma m \vec{v}}$ rel momentum

time part:

$$p^0 = \gamma mc \approx \frac{1}{c}\left(mc^2 + \frac{1}{2}mv^2\right) = \frac{1}{c}\left(mc^2 + K\right) \qquad (21)$$

can identify $\boxed{E_{\mathsf{rel,tot}} = cp^0}$, but then

rest mass has energy $E_{\mathsf{rest}} = mc^2$!

energy, momentum conservation $\to p^\mu$ cons

compact, unified treatment:

$(p^\mu)_{\mathsf{init}} = (p^\mu)_{\mathsf{fin}}$ (4 equations)

The Charms of 4-Momentum

Invariant norm (everyone agrees)

$$p \cdot p = (p^0)^2 - (\vec{p})^2 = E^2 - \vec{p}^2 = m^2 \qquad (22)$$

- rel. (total) energy is $E(p) = \sqrt{(cp)^2 + (mc^2)^2}$
- in rest frame: $\vec{p} = 0 \rightarrow E = mc^2$ "rest mass energy"
- define rel kinetic energy: $K_{\text{rel}} = E - mc^2$
  can show: $K_{\text{rel}} \rightarrow p^2/2m$ if $v \ll c$

Velocity
can show: $\vec{p}/E_{\text{tot}} = \vec{v}$
- non-rel: $Q?$

What if $m = 0$?
- $E^2 - \vec{p}^2 = 0 \rightarrow E = cp$: E is "all kinetic"
- $v = p/E = 1 = c$: moves at c always!

World Lines and Dynamics

for any observer (i.e., any coordinate system):
events along own worldline have

$$(\Delta s)^2 = (\text{observer's apparent elapsed time})^2 \qquad (23)$$

Q: why?

observers' total elapsed time going from events $A \to B$: $\Delta t = \int_a^b ds$

generically: in frame x', elapsed time: $\Delta t = \int_a^b \sqrt{1 - v^2} \, dt'$

consider "race" from event A to event B
accelerated vs non-accelerated ("free") observers
Q: physical picture?
can show: everyone agrees that
non-accelerated observer measures *longest* Δt
Q: this is huge–why? what's special about such observers in SR?

non-accelerated observer $\rightarrow$ no forces

i.e,. a free body!

so in Special Relativity:

of all trajectories from events $A \rightarrow B$

free bodies have max $\int ds$

but free body trajectory is natural motion!

Implications

$\Rightarrow$ free body follows extremum of $\int ds$

   law of motion!

   i.e., variation $\delta \int ds = 0$ selects physical worldline!

$\Rightarrow$ twin "paradox" is not *Q: why?*