Astro 210

Lecture 8

Feb. 2, 2018

Announcements

- **HW2 due online in PDF, Today 5:00 pm**
- HW3 available, due next Friday
- HW1 grading under way; you'll be notified when done
- **register** your iClicker; link on course webpage
- first Planetarium shows Mon Feb 5 and Wed Feb 7
  info online: **reservations**, schedules, directions, report form

Last Time: Sir Isaac Weighs In

Newton's Laws of motion

I. inertia *Q: just a special case of Newton II?*

II. $\vec{F} = m\vec{a}$ *Q: fortunetelling & archæology?*

III. action-reaction

Q: when/where/to what do Newton's laws of motion apply?

Newtonian Universal Gravitation

Q: magnitude of gravity force between masses m *and* M *with distance* r*?*

Q: direction of the force?

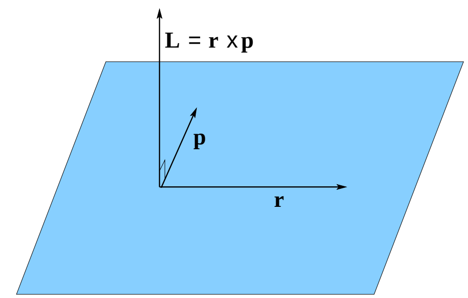
Q: why universal?

Q: what's a variable? what's a constant?

Angular Momentum

For point mass, **angular momentum** defined as:

$$\vec{L} = \vec{r} \times \vec{p} = m\vec{r} \times \vec{v} \tag{1}$$



i.e., using cross product

look at time change:

$$\begin{aligned}
\frac{d}{dt}\vec{L} &= m\dot{r} \times \vec{v} + m\vec{r} \times \dot{v} && (2)\\
&= m\vec{v} \times \vec{v} + m\vec{r} \times \vec{a} && (3)\\
&= \vec{r} \times \vec{F} = \vec{\tau} \quad \text{torque} && (4)
\end{aligned}$$

angular counterpart of Newton II:

- net (linear) force changes linear momentum
- net twisting force = torque changes angular momentum

Gravity and Angular Momentum

angular momentum changed by net torque

$$\frac{d}{dt}\vec{L} = \vec{r} \times \vec{F} \tag{5}$$

when force is due to **gravity**, torque:

$$\vec{\tau} = \vec{r} \times \vec{F} = -G\frac{mM}{r^3}\vec{r} \times \vec{r} = 0 \tag{6}$$

so if force is gravity, then

$$\frac{d}{dt}\vec{L} = 0 \tag{7}$$

and thus $\vec{L} = const$:

angular momentum is **conserved!**

Q: what about gravity force gauranteed this?

What Keeps the Earth in Orbit?

consider *circular orbit* $\rightarrow$ requires centripetal acceleration

angular speed $d\theta/dt = \omega = 2\pi/P = const$

$$\boxed{\vec{a}_c = -\omega^2 \vec{r} = -\frac{v^2}{r}\,\hat{r}}$$

diagram: show $\vec{v}$, $\vec{r}$, $\vec{a}$

Newton II: acceleration demands net force

Newton gravity supplies a force!

$\rightarrow$ Newtonian gravity is crucial and necessary ingredient
for understanding the dynamics of planetary motion
but have to see how the detailed predictions
compare with observation

Testing Newton with Kepler

Program:

- assume Newtonian gravity controls planetary motion
- that is, for any planet let $\vec{F}_{\text{net}} = \vec{F}_{\text{Sun-planet}}$
- input this into Newton's laws
- turn mathematical cranks $\longrightarrow$ predict orbits
- compare predictions with observations *Q: how?*

Solutions: Orbits

For attractive inverse square force law, Newton II orbits are cross sections of cone:

- circle
- ellipse
- parabola
- hyperbola

Circle eccentricity $e = 0$

at each point:

$F = ma = mv_{\mathrm{c}}^2/r$

$\Rightarrow GMm/r^2 = mv_{\mathrm{c}}^2/r$

$\Rightarrow$ circular orbits have speed $\boxed{v_{\mathrm{c}} = \sqrt{\frac{GM}{r}}}$

example: find circular speed 1 AU from Sun

$v_{\mathrm{c}} = 3 \times 10^4$ m/s

Kepler from Newton

Kepler I: Orbits are ellipses
with Sun at one focus

Newton II + Universal Inverse-Square Gravity
bound orbits around a gravitational source M are
- ellipses
- with M at one focus
- eccentricity e set by angular momentum L

Check!

Kepler from Newton

Kepler II: Equal areas in equal times

Newton: consider small time interval dt

move angle $d\theta = \omega dt$

sweep area

diagram: top view: path, $d\theta, \vec{r}, \vec{v}, \vec{v}_\theta$

$$dA = \frac{1}{2} r^2 d\theta = \frac{1}{2} r^2 \omega dt \tag{8}$$

but $\omega = v_\theta / r$, where $\vec{v}_\theta \perp \vec{r}$

$\Rightarrow$ swept area

$$dA = \frac{1}{2} r^2 \frac{v_\theta}{r} dt = \frac{1}{2} r v_\theta dt \tag{9}$$

$\Rightarrow$ swept area

$$dA = \frac{1}{2} r^2 \frac{v_\theta}{r} dt = \frac{1}{2} r v_\theta dt \tag{10}$$

finally, $r v_\theta = |\vec{r} \times \vec{v}| = |\vec{L}|/m$
Q: why?, so

$$dA = \frac{1}{2} \frac{L}{m} dt \tag{11}$$

Woo hoo! were' home free! *Q: why?*

But $L = const$ for radial force ($\vec{r} \times \vec{F} = 0$)
so

$$\frac{dA}{dt} = \frac{L}{2m} = const \tag{12}$$

Kepler II!

Check!

And we learn:
Kepler II comes from angular momentum conservation!

Note: in absence of other forces
and for perfectly spherical Sun
elipses close on themselves–path repeated forever

Kepler III: $a^3 = kP^2$

Newton: can prove generally for elliptical orbits

bad news: price is lotsa algebra

good news: simple to do for circular orbits

circular $\rightarrow r = a$, and $v^2 = GM/a$

but also $v = 2\pi a/P$ *Q: why?*

$$
\begin{aligned}
v^2 = \left(\frac{2\pi a}{P}\right)^2 &= \frac{4\pi^2 a^2}{P^2} && (13)\\
&= \frac{GM}{a} && (14)\\
\Rightarrow\ a^3 &= \left(\frac{GM}{4\pi^2}\right) P^2 && (15)
\end{aligned}
$$

check!

bonus: $k = GM/4\pi^2$ depends on mass of central object

$\rightarrow$ same k for all planets

Energy

For "test" particle m moving due to gravity of M

Gravitational potential energy: *Q: why "potential"?*

$PE = -G\,Mm/r$

Kinetic energy:

$$KE = \frac{1}{2}m\vec{v}^2 = \frac{1}{2}m(v_x^2 + v_y^2 + v_z^2) = \frac{1}{2}m\dot{\vec{r}}^2 \tag{16}$$

Total energy:

$TE = KE + PE = -GMm/r + \frac{1}{2}m\dot{\vec{r}}^2$

key result: $d(TE)/dt = 0$

$\longrightarrow$ total energy conserved!

that is: value of TE the same for all time!

Orbits Revisited

Bound orbits (circle & ellipse): in polar coordinates

$$r(\theta) = \frac{(1 - e^2)a}{1 + e \cos\theta} \tag{17}$$

Circle radius $r = a = const$, eccentricity $e = 0$
recall: circular orbit has constant speed $v_c^2 = GM/r$

$$\begin{aligned} PE &= -\frac{GMm}{r} \quad < 0 & (18) \\ KE &= \frac{1}{2} m v_c^2 = \frac{1}{2} m \frac{GM}{r} = \frac{1}{2} \frac{GMm}{r} = -\frac{1}{2} PE & (19) \\ \Rightarrow TE &= KE + PE = PE/2 = -|PE|/2 < 0 & (20) \end{aligned}$$

$TE < 0$: negative? yes!
Q: what does it mean to have negative energy?

for orbiting system $TE < 0$:

$\rightarrow$ have to *supply* energy to system to break it apart

Why? when particles are *at rest* and "very" *far apart*

$KE = mv^2/2 = 0$

$PE = GMm/r \rightarrow 0$ *Q: how far apart is this?*

and so $TE = KE + PE = 0$: zero total energy

But if start in closed orbits (circular or elliptical): $TE < 0$

$\rightarrow$ To "break" the system from closed orbits, must *add* energy

But energy is conserved $\rightarrow$ not spontaneously added

so system is **bound**

$\Rightarrow$ can't fall apart without external influence

Note: $KE = -PE/2 = |PE|/2$ generally true for gravitating systems in equilibrium:

"virial theorem"

ellipse: semimajor axis a, eccentricity $0 < e < 1$

turns out: TE depends only on a, not e

from conservation of energy

$TE = -GMm/r + \frac{1}{2}mv^2 = -GMm/2a < 0 \rightarrow$ bound

can show

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right) \tag{21}$$

"vis viva" equation ("life force")

discovered prior to concept of energy

handy: gives total speed v at any radius r

Q: at which r is $v = 0$? how does this work for a circular orbit?

Q: for a given orbit (fixed e), when is v max?

Unbound Orbits

Note that both parabolic and hyperbolic orbits are not periodic – do not close on themselves "one-way ticket" past the central object

Parabola

$e = 1$

$$r = \frac{2p}{1 + \cos\theta} \tag{22}$$

p is distance of closest approach

for parabolic orbit:

$TE = 0$ exactly! $\rightarrow$ $KE = -PE$ exactly! very special case!

$\Rightarrow GM/r = \frac{1}{2}v^2$

So at $r = \infty$, $v = 0$

to have this orbit, launch from r with speed

$v_{\text{launch}} = \sqrt{2GM/r}$

iClicker Poll: Orbits

given: test particle m, at distance r from gravitating body M
for test particle to have total energy $TE = 0$
launch from r with speed $v_0 = \sqrt{2GM/r}$

Q: what happens if launch with speed $v > v_0$?

A particle will be in a bound orbit: circle or ellipse

B particle will be unbound, with speed $v \to 0$ as $r \to \infty$

C particle will be unbound, with speed $v > 0$ as $r \to \infty$

Q: why is v_0 a special speed?

Escape Speed

At radius r, define **escape speed** $\boxed{v_{\text{esc}} = \sqrt{2GM/r}}$

- if launch from r with $v_{\text{launch}} < v_{\text{esc}}$
  then $TE < 0$: fall back! (elliptical orbit)
- if launch from r with $v_{\text{launch}} > v_{\text{esc}}$
  then $TE > 0$: escape "easily": $v > 0$ at $r = \infty$
- if launch from r with $v_{\text{launch}} = v_{\text{esc}}$ exactly
  then $TE = 0$ exactly, "just barely" escape

So: escape speed is *minimum speed* needed to leave a gravitating source

Example: escape speed from earth
$v_{\text{esc}} = 11$ km/s = 25,000 mph!
predict the future: if toss object with $v < 25,000$ mph, falls back
but if $v > 25,000$ mph *Q: example?* never returns!