Astro 596/496 NPA

Lecture 8

February 1, 2019

Announcements:

- **Problem Set 1 due today**
- **Preflight 2 due next Friday**
  includes crowdsource group discussion
  and a video to watch and comment!

Last Time: began nuclear reactions

notation: $a(b,c)d$ *Q: which means?*

Q: energy available in a reaction?

Q: how does energy conservation look in a reaction?

Energy Conservation

for *reactions and decays* **total energy is conserved**

$$E_{\text{final}} = E_{\text{initial}} \tag{1}$$

but Einstein says: total energy includes rest mass energy!

example: "2 to 2" reaction $a + b \to c + d$:

$$(m_c + m_d)c^2 + (KE)_f = (m_a + m_b)c^2 + (KE)_i \tag{2}$$

where KE is kinetic energy *in center of mass*

change in kinetic energy:

$$(KE)_f = (KE)_i + (m_a + m_b - m_c - m_d)c^2 \tag{3}$$

Q: implications?

Nuclear Reaction Energy Release Q

reaction kinetic energy change

$$(KE)_f = (KE)_i + (m_a + m_b - m_c - m_d)c^2 = (KE)_i + Q \qquad (4)$$

with reaction Q value:

$$Q = \text{initial masses - final masses} \qquad (5)$$
$$= [m_a + m_b - (m_c + m_d)]\, c^2 \qquad (6)$$
$$= \Delta_a + \Delta_b - \Delta_c - \Delta_c \qquad (7)$$

where last line uses mass defects & baryon conservation

exothermic reactions: $Q > 0$ (mass energy released)
can occur for any $(KE)_i$; "forward" reaction
endothermic reactions: $Q < 0$
need $(KE)_i > |Q|$ to go: " reverse" reaction
$\Rightarrow$ there is a "*threshold*" energy

Reaction Physics

classify reactions according to timescale τ:

Direct: $\tau \simeq t_{\text{cross}} = r_{\text{nuke}}/v$
retain "memory" of initial states
"single step" formation of product
scattered particles most in forward direction
important for high-ish energies $E/A \gtrsim 5$ MeV/nucleon

Compound: $\tau \gg t_{\text{cross}}$
"forget" initial state
form intermediate QM state (resonance)
"compound" nucleus $A + b \rightarrow W \rightarrow c + D$
decays into one of several "channels"
final state particles are isotropic
• back to initial state: elastic scattering
• to one or more new states: reaction!
lower energy, more important for most astro apps

Reaction Rates and Cross Sections

Reaction: $a + b \rightarrow c + d$

Consider particle beam:
"projectiles," number density n_a
incident w/ velocity v
on targets of number density n_b

Due to interactions, targets and projectiles "see" each other as spheres of projected area $\sigma(v)$: the

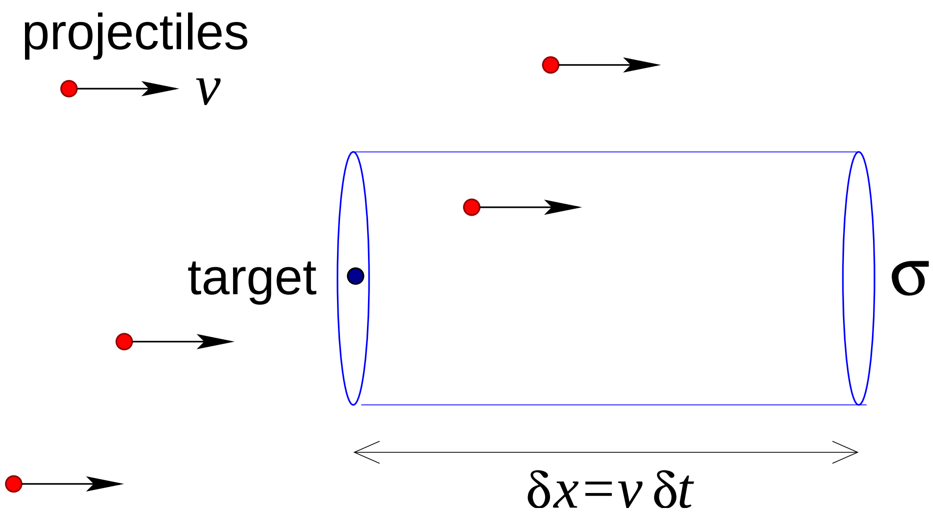
cross section

⋆ fundamental measure interaction strength/probability
⋆ *nuke & particle physics meets astrophysics via* σ

in time δt, what is avg # collisions on one target?
Q: what defines "interaction zone" around target?

interaction zone: particles sweep out "*scattering tube*"

- tube area σ
- length $\delta x = v \delta t$



scattering tube volume around target:

$\delta V = \sigma \delta x = \sigma v \delta t$

collide if a projectile is in the volume

Cross Section, Flux, and Collision Rate

in scattering tube volume $\delta V = \sigma v \, \delta t$,
average number of projectiles in tube $= \mathcal{N}_{\text{proj,tube}} = n_a \delta V$
so: *average number of collisions in δt:*

$$\delta\mathcal{N}_{\text{coll}} = \mathcal{N}_{\text{proj,tube}} = n_{\text{a}} \sigma v \delta t \tag{8}$$

so $\delta\mathcal{N}_{\text{coll}}/\delta t$ gives

$$\text{avg collision rate } \textit{per target } b \quad \boxed{\Gamma_{\text{per}\, b} = n_a \sigma_{ab} v = \sigma_{ab} j_a} \tag{9}$$

where $j_a = n_a v$ is incident flux
Q: Γ units? sensible scalings n_a, σ, v? why no n_b?

Q: average target collision time interval?
Q: average projectile distance traveled in this time?

Reactions: Characteristic Length and Time Scales

estimate *average time between collisions on target b*:
mean free time τ
collision rate: $\Gamma = d\mathcal{N}_{\text{coll}}/dt$
so wait time until next collision set by $\delta N_{\text{coll}} = \Gamma_{\text{per}\, b}\tau = 1$:

$$\tau = \frac{1}{\Gamma_{\text{per}\, b}} = \frac{1}{n_a \sigma v} \tag{10}$$

in this time, projectile a moves distance: **mean free path**

$$\ell_{\text{mpf}} = v\tau = \frac{1}{n_a \sigma} \tag{11}$$

no explicit v dep, but still $\ell(E) \propto 1/\sigma(E)$
Q: physically, why the scalings with n, σ?

∞ *Q: what sets σ for billiard balls?*
Q: what set σ for $e^- + e^-$ scattering?

Cross Section vs Particle "Size"

if particles interact only by "touching"
(e.g., billiard balls)
then $\sigma \leftrightarrow$ particle radii

but: if interact by force field
(e.g., gravity, EM, nuke, weak)
cross section σ *unrelated* to physical size!

For example: e^- has $r_e = 0$ (as far as we know!)
but electrons scatter via Coulomb (and weak) interaction
"touch-free scattering"

Q: what is collision or reaction rate per volume?
Q: what is collision rate per projectile a?

Reaction Rate Per Volume

recall: collision rate *per target b* is $\Gamma_{\text{per}\, b} = n_a \sigma_{ab} v$

total collision rate *per unit volume* is

$$r_{ab} = \frac{dn_{\text{coll}}}{dt} = \Gamma_{\text{per}\, b} n_b = \frac{1}{1+\delta_{ab}} n_a n_b \sigma v \tag{12}$$

Kronecker δ_{ab}: 0 unless particles a & b identical

Note: *symmetric* w.r.t. the two particles

collision rate per projectile a:

$$\Gamma_{\text{per}\, a} = \frac{r_{ab}}{n_a} = \frac{1}{1+\delta_{ab}} n_b \sigma v \tag{13}$$

Note: in general $\Gamma_{\text{per}\, a} \neq \Gamma_{\text{per}\, b}$! *Q: what's going on?*

What if particles have more than one relative velocity?

Reaction Rates: Velocity Distributions

If $v \in$ distribution, rates is average over velocities:

$$\langle r \rangle = \langle n_1 n_2 \sigma v \rangle \tag{14}$$

where $\langle \cdots \rangle$ is avg over relative velocities

given $n_i = \int d^3\vec{v}\ f_i(\vec{v})$,
i.e., $f(v)$ is the velocity distribution of i
and $v_{\rm rel} = |\vec{v}_2 - \vec{v}_1|$

$$\langle r \rangle = \int d^3\vec{v_1} \int d^3\vec{v_2}\ f(\vec{v}_1)\ f(\vec{v}_2)\sigma(v_{\rm rel})v_{\rm rel} \tag{15}$$

Q: (astro)physically relevant velocity distribution(s)?

Thermal Velocity Distribution

Important special case: Non-Relativistic, dilute gas
$\longrightarrow$ distribution $f(\vec{v})$ is Maxwell-Boltzmann

at temperature T

$$f_{\mathrm{MB}}(\vec{v})d^3\vec{v} = n\left(\frac{m}{2\pi kT}\right)^{3/2}\ \exp\left(-\frac{mv^2}{2kT}\right)\left(4\pi v^2 dv\right) \qquad (16)$$

a Gaussian in velocity

So: $r = \int \mathrm{gaussian}_1 \times \mathrm{gaussian}_2$
can show: this reducesto *one* Gaussian
in *relative velocity:*

$$\langle r \rangle = n_1 n_2 \langle \sigma v \rangle \qquad (17)$$

Thermonuclear Reaction Rates

thermal reaction rate coefficient:

average over relative v and reduced mass μ

$$\langle \sigma v \rangle = \left(\frac{\mu}{2\pi kT} \right)^{3/2} \int d^3\vec{v} \; v \; \sigma(v) e^{-\mu v^2/2kT}$$
$$= \sqrt{\frac{8}{\pi\mu}} \frac{1}{(kT)^{3/2}} \int_0^\infty dE \; E \; \sigma(E) e^{-E/kT}$$

"*thermonuclear*" reaction rate

Q: physical significance?

Q: expected behavior at high T? low T?

Thermal Reaction Rates

for nonrelativistic particles a, b at temperature T
the rate of reaction $ab \to cd$ per volume is

$$r_{ab}(t) = n_a n_b \langle \sigma_{ab\to cd} v \rangle_T = \Gamma_{\mathrm{per}\,a}(T)\ n_b = \Gamma_{\mathrm{per}\,b}(T)\ n_a \qquad (18)$$

with thermal reaction rate coefficient

$$\langle \sigma_{ab\to cd} v \rangle_T = \sqrt{\frac{8}{\pi\mu}} \frac{1}{(kT)^{3/2}} \int_0^\infty dE\ E\ \sigma_{ab\to cd}(E)\ e^{-E/kT} \qquad (19)$$

$e^{-E/kT}$ factor suppresses $E \gtrsim kT$

consider **neutron**-nucleus reactions: $a = n\ b = A$
for exothermic reaction, at low energy: $\sigma_{nb\to cd} \propto 1/v$
Q: implications?

contrast **proton** reactions: $a = p\ b = A$
Q: how are things different?

Neutron vs Charged Particle Reactions

neutrons: no Coulomb barrier with nucleus

- low v: de Broglie $\lambda_n = h/m_n v$ increases–n looks "big"!
- and when $\sigma \propto 1/v$: $\sigma v = \langle \sigma v \rangle_T$ is *constant*!

  neutron reaction rates depend weakly on temperature!

protons: Coulomb barrier strong!

$E_C = Z_1 Z_2 e^2/r = Z_1 Z_2$ 1.44 MeV (1 fm$/r$)

Classically, need $\gtrsim 1$ MeV to overcome barrier

charged particle reaction rates depend strongly on temperature!

example: for temperatures like those of center of Sun:

$kT = 0.86$ keV $(T/10^7$ K$) \ll E_C$

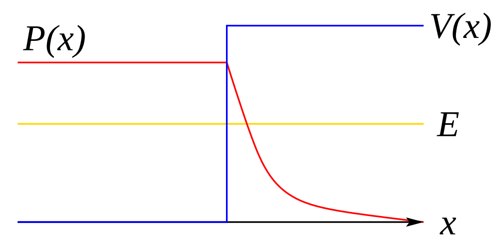
Q: What does this seem to imply?

Q: What's the flaw?

Quantum Mechanics to the Rescure

classically: Coulomb forbid charged particle reactions in Sun
but nuclei are *quantum* particles $\rightarrow$ can tunnel

recall: tunneling under a uniform 1D potential $V(x) = V_0$
by particle with energy $< V_0$:



- wavefunction $\psi(x) = \psi_0 e^{-kx}$
  depth scale $k^2 = 2m(V_0 - E)/\hbar^2$ depends on V_0
- penetration probabilty $P(x) = \|\psi\|^2 = P_0 e^{-2kx}$

Probability to tunnel under Coulomb barrier:

- depth $k^2 \rightarrow k^2(r) = 2m[V(r) - E]/\hbar^2$
- wavefunction $e^{-kx} \rightarrow e^{-\int k(r)\, dr}$

PS2: show that

$$P \propto e^{-2\pi Z_1 Z_2 e^2/\hbar v} = e^{-bE^{-1/2}} \quad (20)$$

Q: behavior?