Astro 596/496 NPA
Lecture 16
February 22, 2019

Announcements:

- **Preflight 3** due today
  kudos on really great answers to the alternative U question!
- **Problem Set 3** out, due next Friday

Last time: particle physics take I

Q: why is the proton stable? the electron? ν?

began cosmic statistical mechanics

for ultra-relativistic bosons b (not conserved, i.e., $\mu_b/T \ll 1$)

Q: number density $n_b(T)$? energy density $\varepsilon_b(T)$? pressure $P_b(T)$?

Q: familiar example?

Ultra-Relativistic Thermal Bosons: Exact Results

for boson species with g internal states (helicity etc)

$$\begin{aligned} n_{\text{rel,b}} &= \frac{g}{2\pi^2\hbar^3c^3}\int_0^\infty dE \frac{E^2}{e^{E/kT}-1} \\ &= g\frac{\zeta(3)}{\pi^2}\left(\frac{kT}{\hbar c}\right)^3 \propto T^3 \end{aligned} \tag{1}$$

$$\begin{aligned} \rho_{\text{rel,b}}c^2 &= \frac{g}{2\pi^2\hbar^3c^3}\int_0^\infty dE \frac{E^3}{e^{E/kT}-1} \\ &= g\frac{\pi^2}{30}\frac{(kT)^4}{(\hbar c)^3} \propto T^4 \end{aligned} \tag{2}$$

where the number density includes a factor of

$$\zeta(3) = \sum_{n=1}^{\infty}\frac{1}{n^3} = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots = 1.20206\ldots \tag{3}$$

Relativistic Bosons: Generalized Blackbody Radiation

Our results generalize the blackbody radiation results:
Planck function for photon number density distribution

$$\frac{dn}{dE} = \frac{g}{2\pi^2\hbar^3c^2}\ E^2\ f(E) = \frac{g}{2\pi^2\hbar^3c^2}\frac{E^2}{e^{E/kT}-1} \tag{4}$$

occupation number $f(E) = 1/(e^{E/kT}-1)$ quanta with E
and energy density distribution

$$\frac{d\varepsilon}{dE} = E\ \frac{dn}{dE} = \frac{g}{2\pi^2\hbar^3c^2}\frac{E^3}{e^{E/kT}-1} \tag{5}$$

where total number and energy densities

$$n = \int_0^\infty \frac{dn}{dE}\ dE \quad \varepsilon = \int_0^\infty \frac{d\varepsilon}{dE}\ dE \tag{6}$$

photons: $g = 2$ polarizations
gives blackbody radiation results
Q: what if anything changes for ultra-relativistic fermions?

Ultra-Relativistic Thermal Fermions

still ultra-relativistic, so $m_f \ll T$ and dimensional analysis same

but for fermions, occupation number limited by Pauli principle
boson expression $f_b = 1/(e^{E/kT} - 1)$ replaced by
fermionic $f_{\rm f} = 1/(e^{E/kT} + 1)$

$$\begin{aligned} n_{\rm rel,f} &= \frac{g}{2\pi^2 \hbar^3 c^3} \int dE \frac{E^2}{e^{E/kT} + 1} & (7) \\ &= \frac{3}{4} n_{\rm rel,b} & (8) \\ \rho_{\rm rel,f} &= \frac{7}{8} \rho_{\rm rel,b} & (9) \end{aligned}$$

so $n \propto T^3$ and $\rho \propto T^4$ for both
e.g., CMB today: $n_{\gamma,0} = 411 \text{ cm}^{-3}$
also $n_{\rm rel,f} < n_{\rm rel,b}$ and $\rho_{\rm rel,f} < \rho_{\rm rel,b}$ (Pauli)

For all ultra-relativistic particles (with $\mu \ll T$):

$$P_{\rm rel} = \frac{1}{3}\rho_{\rm rel} c^2 \tag{10}$$

⋆ holds for *both* fermions and bosons!
e.g., $P_{\rm rel,f} = \rho_{\rm rel,f}/3 < P_{\rm rel,b}$
⋆ shows that relativistic particles have $w_{\rm rel} = +1/3$
⋆ $P \propto T^4$

now imagine a relativistic species with distinct antiparticles so particle X and antiparticle $\bar{X}$ are *different* *Q: examples?*
Q: $n_{\bar{X}}(T)$*?* $\varepsilon_{\bar{X}}(T)$*? pressure* $P_{\bar{X}}(T)$*?*

What About Antiparticles?

particle x and antiparticle $\bar{x}$: both have same mass m
and same internal degrees of freedom g
thus: *for the case we have considered*
all thermal properties should be the same!

$$n_{\bar{X}} = n_X \quad (11)$$

$$\varepsilon_{\bar{X}} = \varepsilon_X \quad (12)$$

$$P_{\bar{X}} = P_X \quad (13)$$

- thermal bath creates $X\bar{X}$ pairs
- not maximally general: really, we *assumed* no net X excess

  if there is, chemical potential $\mu_X \neq 0$ leads to $n_X \neq n_{\bar{X}}$

now consider *the early Universe*
Q: what determines the total radiation density ρ_{rad}*?*

Cosmic Radiation Evolution

for any relativistic species, the *absolute density*
is *set entirely by temperature*: $\rho \propto T^4$!
i.e., particles not conserved, thermal bath adjusts to this

Total cosmic radiation density sums over all relativistic species:

- both particles and antiparticles
- possibly species have different temperatures T_b and T_f

$$\rho_{\rm rel} = \sum_i \rho_{{\rm rel},i} \tag{14}$$

$$= \frac{\pi^2}{30} T^4 \left[\sum_{\rm bosons} g_{\rm b} \left(\frac{T_{\rm b}}{T}\right)^4 + \sum_{\rm fermions} \frac{7}{8} g_{\rm f} \left(\frac{T_{\rm f}}{T}\right)^4 \right] \tag{15}$$

Relativistic Degrees of Freedom

$$\rho_{\rm rel} = \sum_i \rho_{{\rm rel},i} \tag{16}$$

$$= \frac{\pi^2}{30}T^4 \left[\sum_{\rm bosons} g_{\rm b}\left(\frac{T_{\rm b}}{T}\right)^4 + \sum_{\rm fermions}\frac{7}{8}g_{\rm f}\left(\frac{T_{\rm f}}{T}\right)^4\right] \tag{17}$$

$$= g_*(T)\,\frac{\pi^2}{30}T^4 \tag{18}$$

- T is for some reference species, usually photons
- g_* counts "relativistic degrees of freedom"
  e.g., photons contribute $g_{*,\gamma} = 2$
  left-handed $\nu\bar{\nu}$ contributes $g_{*,\nu} = 2 \cdot 7/8 = 7/4$

∞

Particle Census and the Radiation Era

In radiation-dominated early universe:

$$\left(\frac{\dot{a}}{a}\right)^2 \approx 8\pi G \rho_{\rm rel}/3 \propto g_*(T)\ T^4 \tag{19}$$

- ⋆ early expansion history depends on number, types of relativistic particles
- ⋆ microphysics (particle content) of the Universe controls macroscopic cosmic dynamics
- ⋆ ...so any *measure* of early expansion rate is a probe of particle physics!
  ... as we will soon see

Non-Relativistic Statistical Mechanics

Now consider **non-relativistic species** of mass m at T
full results in Director's Cut Extras

order of magnitude:
for non-conserved ultra-relativistic species, we found $n_{\rm rel} \sim \lambda_{\rm deB}(T)^{-3}$

Q: what is thermal de Broglie wavelength here? estimate of n*?*

Q: what is strange about this result?
Hint: what sets $n(T)$*? apply to objects in this room?*

naive estimate of non-relativistic number density:
kinetic energy $\sim kT$ gives momentum $p \sim \sqrt{mkT}$
and *thermal de Broglie wavelength* $\lambda_{\rm deB}(T) \sim \hbar/p \sim h/\sqrt{mkT}$,
and so expect

$$n_{\rm naive}(T) \sim \lambda_{\rm deB}(T)^{-3} \sim \left(\frac{mkT}{\hbar^2}\right)^{3/2} \tag{20}$$

but for a given species m, absolute number density $n(T)$
entirely and universally determined by temperature

but that can't be right!
density of water in you, a beverage, and the air
are all different!

also: for $T = 300$ K this gives
$n_{\rm naive,water} \sim 10^{27}\ {\rm cm}^{-3}$, and $\rho_{\rm naive,water} \sim 3 \times 10^4\ {\rm g/cm}^3$. Yikes!
Q: where did we go wrong?

Non-Relativistic Species: Full Expression

full result includes additional physics we have ignored
derivation appears in Director's Cut Extras

In the *non-relativistic limit* $E(p) \simeq mc^2 + p^2/2m$, $T \ll m$
number density is

$$n = g \left(\frac{mkT}{2\pi\hbar^2} \right)^{3/2} e^{-(mc^2-\mu)/kT} = g \; n_Q \; e^{-(mc^2-\mu)/kT} \tag{21}$$

- where we identify the *quantum concentration*
  $n_Q = (mkT/2\pi\hbar^2)^{3/2} \sim \lambda_{\rm deB}(T)^{-3/2}$
- and where μ is the **chemical potential** or *Fermi energy*

consider the $\mu = 0$ case
Q: $n(T)$ behavior as T lowered in fixed volume?
Q: particle vs antiparticle number density?

Q: so what is the effect of $\mu \neq 0$?

The Chemical Potential

$$n = g\left(\frac{mkT}{2\pi\hbar^2}\right)^{3/2} e^{-(mc^2-\mu)/kT} = g n_Q e^{-(mc^2-\mu)/kT} \tag{22}$$

with $\mu = 0$:

- $n \sim e^{-mc^2/kT} \to 0$ exponentially at low T!
- particle and antiparticle densities are same since mass is same $\to$ thermal bath makes pairs, but $T \ll m$ makes creation rare
- **particle number not conserved!**

when $\mu \neq 0$:

number density can be anything!

chemical potential enforces particle number conservation!

Meaning and Use of the Chemical Potential

in Director's Cut Extras:
μ is free energy cost of adding a particle to the system

if particles *are* conserved: $\mu(T) \neq 0$
$\longrightarrow$ this sets number density implicitly
i.e., $n(T, \mu) = n_{\rm cons}$ sets values of $\mu(T, n_{\rm cons})$

$$\mu = mc^2 - kT \ln \left(\frac{g\ n_Q}{n_{\rm non-rel,cons}} \right) \tag{23}$$

free energy cost μ has rest mass term, plus environment term

in cosmo setting: $n_{\rm non-rel,cons} \propto a^{-3}$, $\rho_{\rm non-rel,cons} \simeq mn \propto a^{-3}$

given number density n: *Q: mass density?*
recall ideal gas: *Q: kinetic energy density? pressure?*

Non-Relativistic Mass Density and Pressure

mass-energy density

$$\rho c^2 \;=\; mc^2\, n + \frac{3}{2}\, kT\, n \simeq \; \varepsilon_{\text{rest mass}} = mc^2\; n \tag{24}$$

pressure

$$P \;=\; \frac{2}{3}\varepsilon_{\text{kinetic}} = nkT \;\ll\; \rho c^2 \tag{25}$$

Note:

- recover ideal gas law!
- $P \ll \rho c^2$, and so *equation of state parameter* is

$$w_{\text{non-rel}} = \frac{P}{\rho c^2} \approx \frac{kT}{mc^2} \approx 0$$

BBN: Baryons vs Photons

BBN key controlling parameter:
baryon-to-photon ratio by number
define $\boxed{n_B/n_\gamma \equiv \eta}$

Q: why should photons affect abundances?

Q: individually, do n_B *and* n_γ *depend on* T*? How?*

Q: so how does η *change over time?*

The Baryon-to-Photon Ratio

photons are important for BBN in several ways

- *macroscopically*: photons critical to cosmic *dynamics*
  photons contribute to radiation density ρ_{rel}
  that dominates expansion rate $H^2 \sim G\rho_{\text{rad}}$
- *microscopically*: energetic photons can unbind nuclei
  e.g., $\gamma d \to np$
  competes with nuke reactions, impedes element formation!

photons: $n_\gamma \propto T^3$ (relativistic thermal distribution)
baryons: $n_B \propto a^{-3} \propto T^3$ (baryon # conservation)
$\Rightarrow$ baryon-to-photon ratio $\eta = n_B/n_\gamma \sim (aT)^{-3} \sim const$
η *same today as at end of BBN!*

Baryon-to-Photon Ratio: Spoiler Alert

Predicted primordial abundances depend on η
$\rightarrow$ can use observations to *measure* baryon/photon ratio
i.e., can us BBN to *find* η as *output*

but preview: will find

$$\eta \sim 10^{-9}$$

so $n_\gamma > 10^9 n_B$ thermal (CMB) photons per cosmic baryon!

this imbalance has huge implications!

BBN: Pioneering Days

Gamow group: (Gamow, Alpher, Herman; 1940's)
Initial conditions:
early U → high density → all neutrons
(like neutron star)
`www:` $\alpha\beta\gamma$ `paper`

Hayashi (1950):
weak interactions non-negligible
weak equilibrium determines n/p ratio

Alpher, Follin, & Herman (1953):
first "modern" calculation of n/p ratio

Burbidge, Burbidge, Fowler & Hoyle ($\equiv$ B^2FH, 1957):
heavy elements made in stars (not BB)
outlined nuclear processes light elements: unknown "X-process"

Director's Cut Extras

Cosmic Archaelogy: The Early Universe

When are high-energy processes/particles abundant?

- Universe has temperature now: CMB $T_0 = 2.725$ K
  $\Rightarrow$ cosmic matter was once in thermal equilibrium
- in thermal bath, typical particle energy is $E \sim kT$
- cosmic temperature $T \propto 1/a = 1 + z$

Therefore:

- when primordial soup at high-E $\rightarrow$ high T $\rightarrow$ early times

★ the early universe is the realm of particle physics

★ cosmic *particle* history $\Leftrightarrow$ cosmic *thermal* history

Cosmic Statistical Mechanics

Consider a "gas" of quantum particles (massive or massless) states "smeared out" around classical $\vec{x}$ and $\vec{p}$ values

define **occupation number** or **distribution function** $f(\vec{x},\vec{p})$: number of particles in each phase space "cell":

$$dN = g f(\vec{x},\vec{p}) \frac{d^3\vec{x}\ d^3\vec{p}}{(2\pi\hbar)^3} = g f(\vec{x},\vec{p}) \frac{dV_{\rm space}\ dV_{\rm momentum}}{(2\pi\hbar)^3} \qquad (26)$$

where g counts internal (spin/helicity) degrees of freedom and $dx\,dp/2\pi\hbar$ counts # of quantum states per cell

particle phase space occupation f determines bulk properties
Q: how? Hint–what's # particles per unit spatial volume?

for a given spatial volume element $dV_{\text{space}} = d^3\vec{x} = dx\,dy\,dz$
number per unit (spatial) volume–i.e., **number density**–is

$$dn = \frac{dN}{d^3\vec{x}} = g f(\vec{x}, \vec{p}) \, \frac{d^3\vec{p}}{(2\pi\hbar)^3} \tag{27}$$

$\rightarrow$ f gives *distribution* of momenta at each spatial point

Q: what's f for gas of (classical) particles all at rest?
Q: f for a (classical) particle beam–directed, monoenergetic?
Q: what's f for (classical) harmonic oscillator?

Q: given f, how to formally compute bulk properties n, ε, P?

Number density

$$n(\vec{x}) = \frac{d^3N}{d^3x} = \frac{g}{(2\pi\hbar)^3}\int d^3\vec{p}\ f(\vec{p},\vec{x}) \tag{28}$$

Mass-energy density

$$\varepsilon(\vec{x}) = \rho(\vec{x})c^2 = \langle E\rangle\ n = \frac{g}{(2\pi\hbar)^3}\int d^3\vec{p}\ E(p)\ f(\vec{p},\vec{x}) \tag{29}$$

Pressure

$$P(\vec{x}) = \langle p_i v_i\rangle_{\text{direction}\,i}\ n = \frac{\langle pv\rangle}{3} n = \frac{g}{(2\pi\hbar)^3}\int d^3\vec{p}\ \frac{p\,v(p)}{3}\ f(\vec{p},\vec{x}) \tag{30}$$

Q: these expressions are general–simplifications due to cosmo principle?

FRLW universe:

- homogeneous $\longrightarrow$ no $\vec{x}$ dep
- isotropic $\longrightarrow$ only $\vec{p}$ magnitude important $\longrightarrow$ $f(\vec{p}) = f(p)$

in **thermal equilibrium**:

▷ Boson occupation number is **Bose-Einstein** dist'n

$$f_{\rm b}(p) = \frac{1}{e^{(E-\mu)/kT} - 1} \quad (31)$$

▷ Fermion occupation number is **Fermi-Dirac** dist'n

$$f_{\rm f}(p) = \frac{1}{e^{(E-\mu)/kT} + 1} \quad (32)$$

Note: μ is "chemical potential" or "Fermi energy"
$\mu = \mu(T)$ but is *independent* of E

If $E = E_{\rm tot}, \mu \gg T$: both $\longrightarrow$ $f = e^{-(E-\mu)/kT} \ll 1$
$\longrightarrow$ **Boltzmann distribution**

Chemical Potential & Number Conservation

For a particle species in thermal equilibrium

$$f(p;T,\mu) = \frac{1}{e^{(E-\mu)/kT} \pm 1} \tag{33}$$

What is μ, and what does it mean physically?

First, what if $\mu = 0$
then f, n, P depend only on T
$\rightarrow$ everything at same T has same ρ, P!
sometimes true! *Q: examples?* but not always!

but n often conserved
$\rightarrow$ fixed by initial conditions, not T
$\rightarrow$ if particle number conserved, μ determined
by solving $n_{\text{cons}} = n(\mu, T) \rightarrow \mu(n_{\text{cons}}, T)$
so: $\boxed{\mu \neq 0 \Leftrightarrow \text{particle number conservation}}$

if "chemical" equilibrium:

- reactions change particle numbers among species
- equilibrium: forward rate = reverse rate

$a + b + \cdots \leftrightarrow A + B + \cdots$

then

$$\sum_{\text{initial particles} i} \nu_i \ \mu_i = \sum_{\text{final particles} f} \nu_i \ \mu_f \tag{34}$$

where ν_i is number of particles of type i in the reaction
i.e., the *Stoichiometric coefficient* for particle i

so *in chemical equilibrium*:
sum of chemical potentials "conserved"

Equilibrium Thermodynamics

Gas of particles: mass m, temperature T:
n, ρ, and P in general complicated
because of relativistic expression $E(p) = \sqrt{p^2 + m^2}$
but simplify in ultra-rel and non-rel limits
$\longrightarrow$ controlled by m vs T comparison

Non-Relativistic Species

$E(p) \simeq mc^2 + p^2/2m$, non-rel: $T \ll m$
for $\mu \ll T$: Maxwell-Boltzmann, same for Boson, Fermions

for non-relativistic particles = *matter*
energy density, number density vs T?

Non-Relativistic Species: Cosmic Matter

In the limit $E(p) \simeq mc^2 + p^2/2m$, $T \ll m$

$$n = g \left(\frac{mkT}{2\pi\hbar^2}\right)^{3/2} e^{-(mc^2-\mu)/kT} \qquad (35)$$

$$\rho c^2 = mc^2\, n + \frac{3}{2}\, kT\, n \simeq \varepsilon_{\text{rest mass}} = mc^2\, n \qquad (36)$$

$$P = \frac{2}{3}\varepsilon_{\text{kinetic}} = nkT \ll \rho c^2 \qquad (37)$$

Note:

- recover ideal gas law!
- $P \ll \rho c^2 \rightarrow w_{\text{non-rel}} \ll 1 \approx 0$
- if particles *not* conserved: $\mu = 0$
  Q: behavior of $n(T)$? why isn't this crazy?
- if particles are conserved: $\mu(T) \neq 0$

$\rightarrow$ this sets number density implicitly
i.e., $n(T,\mu) = n_{\text{cons}}$ sets values of μ
in cosmo setting: $n_{\text{non-rel,cons}} \propto a^{-3}$, $\rho_{\text{non-rel,cons}} \simeq mn \propto a^{-3}$

Ultra-Relativistic Species: Cosmic Radiation

take limit $E(p) \simeq cp \gg mc^2$ (i.e., $kT \gg mc^2$)

Also take $\mu = 0$ ($\mu \ll kT$)

note: now contributions from states with $E, \mu \ll T$
expect bosons, fermions $\rightarrow$ different n, ρ, P for same T
Q: why? Hint–think about form of $f_{\rm b}$ *and* $f_{\rm f}$
Q: which particle type should have larger n, ρ, P *at fixed* T*?*

energy density, number density?
Q: you know this already for bosons!

for relativistic bosons

$$n_{\rm rel,b} = g\frac{\zeta(3)}{\pi^2}\left(\frac{kT}{\hbar c}\right)^3 \propto T^3$$
$$\rho_{\rm rel,b}c^2 = g\frac{\pi^2}{30}\frac{(kT)^4}{(\hbar c)^3} \propto T^4$$

where

$$\zeta(3) = \sum_{n=1}^{\infty}\frac{1}{n^3} = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots = 1.20206\ldots \qquad (38)$$

relativistic fermions:

$$n_{\rm rel,f} = \frac{3}{4}n_{\rm rel,b} \qquad (39)$$
$$\rho_{\rm rel,f} = \frac{7}{8}\rho_{\rm rel,b} \qquad (40)$$

so $n \propto T^3$ and $\rho \propto T^4$ for both

e.g., CMB today: $n_{\gamma,0} = 411\ \text{cm}^{-3}$

also $n_{\rm rel,f} < n_{\rm rel,b}$ and $\rho_{\rm rel,f} < \rho_{\rm rel,b}$ (Pauli)

Chemical Potential: Thermodynamic Definition

consider a system where we can adjust the volume V
and with internal energy E, but fixed particle number N
then entropy S chages as

$$T\ dS = dE + p\,dV \quad \text{fixed } N \tag{41}$$

now *allow N to change* – this also changes entropy

$$T\ dS = dE + p\,dV - \mu\,dN \tag{42}$$

we see that the *chemical potential* is

$$\mu(S,V) = \left(\frac{\partial E}{\partial N}\right)_{S,V} \tag{43}$$

measures change in interal energy due to adding a particle
adiabatically and at constant volume

define *Helmholtz free energy* $F = E - TS$, then

$$dF = -p\,dV - S\,dT + \mu\,dN \tag{44}$$

then can also define chemical potential as

$$\mu(T,V) = \left(\frac{\partial F}{\partial N}\right)_{T,V} \tag{45}$$

free energy cost of adding one particle at constant T and V

for non-relativistic gas with conserved density $n_{\rm cons}$ we found

$$\mu = mc^2 - kT \ln\left(\frac{g\ n_Q}{n_{\rm cons}}\right) \tag{46}$$

free energy particle cost dominated by *rest mass* mc^2
but is *reduced slightly by "environmental" term* $kT \ln(g n_Q/n)$

Note: if $\mu = 0$ then no free energy cost to adding particles
$\rightarrow$ particle number can change freely $\rightarrow$ no particle conservation